

Lectures on the Routh-Hurwitz problem

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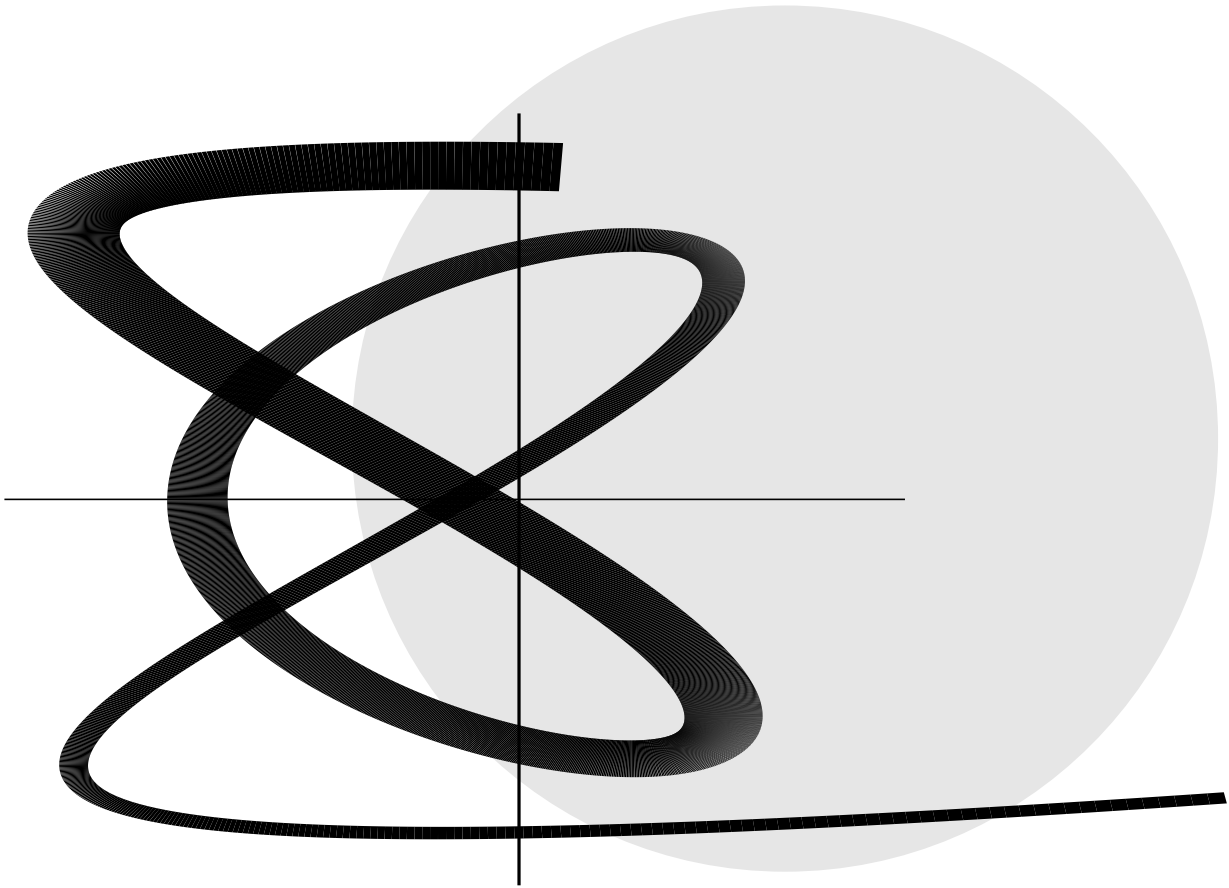
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CHAPTER 1

Basic theory



"Well", said Owl, "the customary procedure in such cases is as follows."

A. A. Milne, WINNIE-THE-POOH.

Introduction

In the vicinity of the zero fixed point¹ of the differential equation

$$\dot{x} = Ax + o(x) \quad (x \in \mathbb{R}^n), \quad (1)$$

the behaviour of other solutions is mostly determined² by the location of the eigenvalues of the matrix A with respect to the imaginary axis. For instance, if the spectrum of the matrix A lies in the open left half-plane of the complex plane, then the fixed point is asymptotically stable. It is unstable if the matrix A has *at least* one eigenvalue with positive real part. If we know that the matrix A has no purely imaginary eigenvalues, then the local structure of the phase portrait of the system (1) can be determined by the number of eigenvalues of the matrix A in the left and right half-planes of the complex plane. For all these reasons, matrix theory methods and techniques for answering such questions are of great interest in stability theory. Since the spectrum of the matrix A is the set of all roots of its characteristic polynomial $p(\lambda) \equiv \det(\lambda I - A)$, then the same questions are to be answered in the theory of polynomials, too.³

A polynomial p is called *stable* if all its roots lie in the open left half-plane. The *Routh-Hurwitz problem* consists in finding conditions of polynomial stability⁴ and, generally, in the study of those properties of the polynomial that are in some way connected with location of its roots with respect to the imaginary axis.

The Hurwitz criterion is traditionally viewed as the main result on stable polynomials. It will be discussed in §6. The practical use of this theorem is usually limited, in the context of direct computations, to polynomials of low degrees (3rd, 4th, or 5th). In fact, the Hurwitz criterion is only one of the facts of a compact algebraic theory – a theory that contains other practically useful results, that is related to important and interesting chapters of algebra and analysis, and, finally, that is beautiful. The last point is important. It is most difficult to master the art of producing mathematical results, of posing and solving problems, but it is an art worth learning⁵. Unexpected ideas, subtle arguments are seldom fruits of pure imagination; more often, they are results of observation, perseverance, and good taste of their author. That kind of experience comes with learning things that are worth emulating. The theory of stable polynomials provides a great sample of this kind. Within this theory, everyday mathematical notions and ideas interact, reshape themselves, and bring about new realms of possible applications. Taking these didactic ideas to heart, the author did not intend to simply give a standard list of facts, but instead to show the development of this mathematical theory, so that the

¹Lit.: point of equilibrium [translators' remark].

²In the so-called *critical case*, the nonlinear term $o(x)$ «gets voting rights» and influences the behaviour of solutions in an arbitrarily small vicinity of the fixed point.

³The transition from a matrix to its characteristic polynomial is far from being harmless. Firstly, matrix properties that may have an influence on its spectrum may be lost or hidden as a result. For example, it is easy to establish that all eigenvalues of a symmetric matrix must be real, but it is more difficult to understand how this property affects the coefficients of its characteristic polynomial. Secondly, the characteristic polynomial is useful only for general theoretical questions and does not easily submit to numerical computations and analytical derivations.

⁴Both terms became customary but they are rather unsatisfactory. The former is not good because there is a «stable» fixed point of a system of differential equations not of a polynomial. The latter is also bad, since the first person who posed the «Routh-Hurwitz problem» and who obtained fundamental results in this area was in fact C. Hermite. Here is the chronology of works: C. Hermite — 1856, E. J. Routh — 1877, A. Hurwitz — 1895.

⁵The reader is referred to textbooks [23]–[24] and problem book [25, 26].

reader may become a participant in its re-creation. Acknowledging that the interest of some student readers may be quite pragmatic, the author at the same time tried to separate the basic material, which one ought to learn in any case and which is presented very tersely, from discussions and additional points made in remarks, problems, footnotes etc. Incidentally, one can learn to apply the Hurwitz theorem by solving the following fairly typical problem.

Problem. Find all fixed points of the Lorentz system⁶

$$\dot{X} = \sigma Y - \sigma X, \quad \dot{Y} = rX - Y - XZ, \quad \dot{Z} = XY - bZ. \quad (2)$$

(σ, r, b are positive parameters). Get to know the statement of Hurwitz' theorem in § 6 and apply it to investigate the stability of the fixed points found.

A solution to this problem is given in the Appendix. It is however recommended that the reader obtains this solution on her/his own or at least tries to do so.

Acknowledgment

The author thanks Mikhail Tyaglov, who pointed out a number of typos in the first version of these notes; those typos are corrected in the present version. The author is also grateful to Olga Holtz for her active interest in these notes and assistance with their publication.

1. Stodola condition

One of the most basic but rather useful facts on stable polynomials is contained in the following theorem, which is usually attributed to the Slovak engineer A. Stodola (1893).

Theorem 1 (Stodola). *If a polynomial with real coefficients is stable, then all its coefficients are of the same sign.*

PROOF. The roots of a real polynomial are symmetric with respect to the real axis. Let

$$p(z) = a_0 \prod_j (z - \lambda_j) \cdot \prod_k (z - \alpha_k - i\beta_k)(z - \alpha_k + i\beta_k),$$

where λ_j are the real and $\alpha_k \pm i\beta_k$ are the nonreal roots of the polynomial p (note that $\lambda_j, \alpha_k < 0$). Since the binomials $z - \lambda_j$ and the trinomials $z^2 - 2\alpha_k z + (\alpha_k^2 + \beta_k^2)$ have positive coefficients, their product has the same property. \square

The role of Theorem 1 is quite clear: it provides a very easily verified *necessary* condition of polynomial stability. It cannot be reversed, except for very low degrees:

Problem 2. *A quadratic polynomial with positive coefficients is stable.*

In general, the following partial converse holds:

Problem 3. *A polynomial of degree n with positive coefficients has no roots in the sector*

$$|\arg z| \leq \frac{\pi}{n}. \quad (3)$$

Hint: Consider a broken line with $n+1$ segments whose k th segment is parallel to the vector $a_k z^{n-k}$ ($k = 0, \dots, n$). If $\arg z$ is too small, this broken line cannot be closed.

⁶This system is related to one of the classical hydrodynamical problems, viz., the onset problem for convectional motion in a fluid horizontal layer heated from below. The Lorentz system is interesting due to the fact that its trajectories have very complex behaviour for certain values of parameters.

Problem 4. A polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z - a_n$ ($a_0, \dots, a_n > 0$) has exactly one root on the positive half-axis; it is smaller than the absolute value of any other root of p .

THE QUESTION HOW THE SIGNS OF THE COEFFICIENTS AFFECT THE ROOT DISTRIBUTION OF A POLYNOMIAL IS QUITE INTERESTING *per se* (SEE [11, part V, Chapter I]), BUT LEADS AWAY FROM THE TOPIC OF STABLE POLYNOMIALS. ON THE OTHER HAND, THE CONSTRUCTIONS OF OUR NEXT SECTION TURN OUT TO BE VERY FRUITFUL.

2. Nyquist-Mikhailov hodograph

Let $p(z)$ be a polynomial⁷ of degree n . In the complex plane \mathbb{C} , consider the curve⁸

$$\Gamma_p \equiv \{ i^{-n} p(i\omega) : \omega \in \mathbb{R} \}. \quad (4)$$

As the parameter ω runs from $-\infty$ to ∞ , the curve is traversed in a certain direction. This oriented curve is called the Nyquist-Mikhailov⁹ *hodograph*¹⁰, or simply the hodograph, or the amplitude-phase characteristic of the polynomial p .

Assume that the polynomial p has no roots on the imaginary axis. In this case, Γ_p does not go through zero and the function

$$\varphi_p(\omega) \equiv \text{Arg } i^{-n} p(i\omega) = \text{Im Log } i^{-n} p(i\omega) \quad (\omega \in \mathbb{R}) \quad (5)$$

is *continuous* at each point of the real axis. Note that this function is defined up to an additive constant of the form $2\pi k$ ($k \in \mathbb{Z}$), and its values do not have to lie in the interval $[0, 2\pi]$. In the sequel, we will be interested in the increment¹¹

$$\Delta_p \equiv \varphi_p \Big|_{-\infty}^{+\infty}, \quad (6)$$

which is defined unambiguously.

Lemma 5. If $p(z) = z - \lambda$ ($\text{Re } \lambda \neq 0$), then $\Delta_p = -\pi \text{ sign Re } \lambda$.

PROOF. The hodograph Γ_p is a horizontal line traversed from left to right that intersects the imaginary axis at the point $i \text{Re } \lambda$. Obviously, as ω runs from $-\infty$ to $+\infty$, the radius-vector of a point on the hodograph makes a clockwise turn of magnitude π if $\text{Re } \lambda > 0$ (counter-clockwise if $\text{Re } \lambda < 0$). \square

Theorem 6 (Hermite). If a polynomial p has n_- roots in the left half-plane and n_+ roots in the right half-plane but no roots on the imaginary axis, then

$$\Delta_p = \pi(n_- - n_+). \quad (7)$$

PROOF. If $p = a_0 p_1 \cdots p_n$, where $p_k(z) = z - \lambda_k$, then

$$\varphi_p = \arg a_0 + \varphi_{p_1} + \cdots + \varphi_{p_n} \quad \text{and} \quad \Delta_p = \Delta_{p_1} + \cdots + \Delta_{p_n}.$$

⁷For now, we do not need to assume that the coefficients of the polynomial are real, although this is indeed the case in most applications.

⁸The normalizing factor i^{-n} is not of vital importance. It is needed to simplify some formulæ of the next sections.

⁹Nyquist's name was added in translation [translators' remark].

¹⁰The work of A. V. Mikhailov (1937) as well as the earlier work of the American engineer H. Nyquist (1932) attracted attention to the geometrical method that we describe here. Especially important applications of this method were found in automatic control theory, mostly thanks to papers of the Romanian mathematician V. M. Popov. However, «Mikhailov's hodograph» was discovered by C. Hermite.

¹¹Along the way, we will also prove that the limits $\varphi_p(\pm\infty)$ exist.

From Lemma 5, it follows that

$$\Delta_p = -\pi (\operatorname{sign} \operatorname{Re} \lambda_1 + \cdots + \operatorname{sign} \operatorname{Re} \lambda_n) = \pi(n_- - n_+).$$

□

Remark 7. Since there are no roots on the imaginary axis, we have $n_- + n_+ = \deg p$. Together with (7), this enables us to find both numbers n_- and n_+ .

Remark 8. The increment Δ_p achieves its maximal value, which equals $\pi \deg p$, for stable polynomials.

Problem 9. If a polynomial p is stable, prove that φ_p is monotone increasing on \mathbb{R} .

Problem 10. Let γ be a closed oriented curve on the Riemann sphere (e.g., the imaginary axis is such a curve). Let us introduce the «generalized hodograph» $\Gamma_p^\gamma \equiv \{i^{-n}p(\omega) : \omega \in \gamma\}$ and define the quantities φ_p^γ and Δ_p^γ analogously to (5) and (6). Consider the unit circle and the sector (3) and formulate for them the analogue of Theorem 6.

Problem 11. Theorem 6 is related to the «argument principle» in the theory of analytic functions. What is the value of the integral $\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{p'(z)}{p(z)} dz$, taken over the imaginary axis in the sense of the Cauchy principal value?

Problem 12. Investigate the hodograph of a rational function. Which rational functions do you think should be called «stable»?

Problem 13. On the front page you see a stylized picture of the hodograph of the polynomial $p(z) = 32z^6 + 12z^5 + 46z^4 + 21z^3 + 16z^2 + 7z + 1$ (the thickness of the curve decreases as the parameter ω increases). Where are the roots of $p(z)$ located with respect to the imaginary axis? Using MAPLE, draw this curve on your own. Try to change one of the coefficients of $p(z)$. What happens with the hodograph?

THEOREM 6 OBTAINED BY SUCH SIMPLE TOOLS CAN ALREADY BE APPLIED TO COUNT THE NUMBER OF ROOTS OF THE POLYNOMIAL p TO THE LEFT AND TO THE RIGHT OF THE IMAGINARY AXIS. WE CAN ENTRUST THE DRAWING OF HODOGRAPHS TO A COMPUTER AND DETERMINE THE NUMBER OF HALF-TURNS VISUALLY.

IT IS WORTH CONSIDERING THE FOLLOWING QUESTIONS: WHY NOT USE A COMPUTER TO COUNT ALL ROOTS OF A POLYNOMIAL AS WELL? WHICH OF THE TWO PROBLEMS WILL REQUIRE MORE CALCULATIONS? FOR WHICH POLYNOMIALS WILL THE PERTINENT CALCULATIONS BE HARD AND THEIR RESULTS UNRELIABLE? WHICH PROBLEMS MAY ARISE FOR A DEVELOPER AND FOR A USER OF SUCH A PROGRAM?

MOST LIKELY, WE WILL COME TO THE CONCLUSION THAT IT IS PREMATURE TO WRITE A PROGRAM, AND THE «SPECIFIC» QUESTION OF COUNTING THE NUMBER Δ_p OF HALF-TURNS REQUIRES A MATHEMATICAL RATHER THAN A PROGRAMMING SOLUTION. THIS IS INDEED THE CASE.

3. Cauchy indices

The quantity Δ_p characterizes quite general topological properties of the curve Γ_p . It turns out that for counting Δ_p it is enough to know in which order a point moving along Γ_p crosses the coordinate half-axes.

Let

$$p(z) \equiv a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0 \in \mathbb{R}, a_0 > 0). \quad (8)$$

Let us consider the real polynomials

$$\begin{aligned} f_0(\omega) &\equiv +\operatorname{Re} [i^{-n}p(i\omega)] = a_0\omega^n + \dots, \\ f_1(\omega) &\equiv -\operatorname{Im} [i^{-n}p(i\omega)] = (\operatorname{Re} a_1)\omega^{n-1} + \dots \end{aligned} \quad (9)$$

satisfying

$$i^{-n}p(i\omega) = f_0(\omega) - if_1(\omega), \quad \deg f_1 < \deg f_0. \quad (10)$$

If all coefficients of the polynomial (8) are real, then we have:

$$\begin{aligned} f_0(\omega) &\equiv a_0\omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - \dots, \\ f_1(\omega) &\equiv a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - \dots. \end{aligned} \quad (11)$$

Now it is time to discuss the assumption of the previous section that the polynomial p has no roots on the imaginary axis. How can we check this condition? From (10) one can see that, for $\omega \in \mathbb{R}$,

$$p(i\omega) = 0 \Leftrightarrow f_0(\omega) = f_1(\omega) = 0 \Leftrightarrow \gcd(f_0, f_1)(\omega) = 0. \quad (12)$$

Thus, we need to use the Euclidean algorithm to find the greatest common divisor $d \equiv \gcd(f_0, f_1)$. In the simplest case we get $d = 1$. Otherwise, we need to find out whether the polynomial d has real roots.

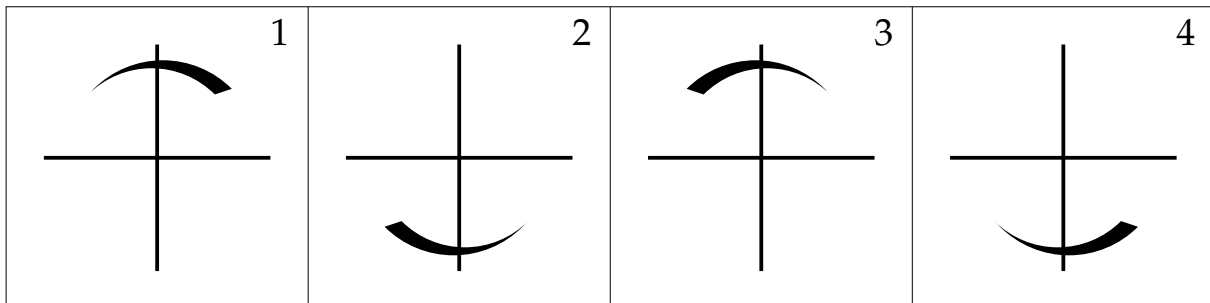
Remark 14. As we will see later, the idea of using the Euclidean algorithm is extraordinarily fruitful. Now we simply ran into it and risk to pass it by, not noticing that it is key to solving the entire problem. Can we at this stage *guess*, perhaps only *feel*, the value of this accidental idea to develop it afterwards?

Assume that the polynomial p is continuously perturbed so that, at some moment, one or several of its roots intersect the imaginary axis, thus changing the values of n_- and n_+ , which we are interested in. If we apply the Euclidean algorithm to the corresponding polynomials f_0 and f_1 , then its *final result*, the greatest common divisor d , will forget what happened. Most likely, d was equal to 1 and will again become equal to 1. But what if the memory of those events will be preserved in the *by-product* of the algorithm, which are usually thrown out as useless? Later we will see that this guesswork will be confirmed in its entirety.

Still assuming that the curve Γ_p does not go through zero, let us now consider how it intersects the imaginary axis. Let $\omega_0 < \omega_1 < \dots < \omega_m$ be the values of the parameter ω for which the intersections occur. According to (9), ω_k are real roots of the polynomial f_0 of odd multiplicity (so typically simple). Denote

$$i_k \equiv \lim_{\omega \rightarrow \omega_k} \operatorname{sign} \frac{d}{d\omega} \varphi_p(\omega) \quad (k = 0, 1, \dots, m). \quad (13)$$

In other words, $i_k = -1$ ($i_k = +1$) if the radius-vector of a point on the hodograph turns clockwise (counter-clockwise) when ω passes through the point ω_k (see the picture).



Lemma 15. $\varphi_p \Big|_{\omega_{k-1}}^{\omega_k} = \frac{\pi}{2} (i_{k-1} + i_k) \quad (k = 1, \dots, m)$

PROOF. For definiteness, suppose that ω_{k-1} corresponds to an intersection of type 1 on the picture. Then the next value ω_k corresponds to an intersection of type 2 or 3. In the former case, $\varphi_p \Big|_{\omega_{k-1}}^{\omega_k}$ equals $+\pi$, in the latter case, zero, which agrees with the statement of the lemma. The remaining three possibilities can be considered similarly. \square

Lemma 16. $\varphi_p \Big|_{-\infty}^{\omega_0} = \frac{\pi}{2} i_0, \quad \varphi_p \Big|_{\omega_m}^{+\infty} = \frac{\pi}{2} i_m$

PROOF. Since $\deg f_0 > \deg f_1$, we have

$$\tan \varphi_p(\omega) = -\frac{f_1(\omega)}{f_0(\omega)} \rightarrow 0 \quad (\omega \rightarrow \pm\infty).$$

Consequently, the limit directions of the radius-vector are horizontal.

For definiteness, suppose that the radius-vector approaches the direction of the positive real half-axis when $\omega \rightarrow -\infty$. Then between $-\infty$ and ω_0 there must be an intersection of type 2 or 3 (see the picture). In the former case $\varphi_p \Big|_{-\infty}^{\omega_0}$ equals $+\frac{\pi}{2}$. In the latter case it is equal to $-\frac{\pi}{2}$. This agrees with the statement of the lemma. The remaining possibilities must be considered similarly. \square

Lemma 17. *If p has no roots on the imaginary axis, then*

$$\Delta_p = \pi (i_0 + i_1 + \dots + i_m) \tag{14}$$

PROOF. The increment is additive:

$$\Delta_p = \varphi_p \Big|_{-\infty}^{\omega_0} + \varphi_p \Big|_{\omega_0}^{\omega_1} + \dots + \varphi_p \Big|_{\omega_{m-1}}^{\omega_m} + \varphi_p \Big|_{\omega_m}^{+\infty}.$$

Therefore, the application of Lemma 15 and Lemma 16 implies (14).

Strictly speaking, we should also consider the case $m = -1$, i.e., the case when the hodograph does not intersect the imaginary axis at all. Using the same reasoning as in the proof of Lemma 16, we then will see that $\Delta_p = 0$. This is indeed the proper way to understand the formula (14) in case $m = -1$. \square

The quantities i_k and their sum occur in other applications. They have specific names. Let us consider the rational function

$$R(\omega) \equiv \frac{f_1(\omega)}{f_0(\omega)} \quad (\deg f_1 < \deg f_0), \tag{15}$$

where the numerator and the denominator are arbitrary real polynomials and are not necessarily determined from (9).

Let $\omega_0 < \omega_1 < \dots < \omega_m$ be the real poles of R of odd order. This means that $R(\omega)$ changes its sign as it «goes through ∞ » when ω goes through ω_k .

The quantity

$$\text{Ind}_{\omega_k}(R) \equiv \begin{cases} +1, & \text{if } R(\omega_k - 0) < 0 < R(\omega_k + 0), \\ -1, & \text{if } R(\omega_k - 0) > 0 > R(\omega_k + 0) \end{cases} \tag{16}$$

is called the *index* of the function R at its real pole ω_k of odd order.

The quantity

$$\text{Ind}_a^b(R) \equiv \sum_{k: a < \omega_k < b} \text{Ind}_{\omega_k}(R) \quad (17)$$

is called the *Cauchy index* of the function R on the interval (a, b) .

A comparison with (13) and with the picture now shows that

$$i_k = \text{Ind}_{\omega_k}(R), \quad \text{where} \quad R \equiv \frac{f_1}{f_0}.$$

Theorem 18. *Let the polynomial (8) have n_- roots in the left half-plane, n_+ roots in the right half-plane, and no roots on the imaginary axis. Then*

$$n_- - n_+ = \text{Ind}_{-\infty}^{+\infty} \left(\frac{f_1}{f_0} \right),$$

where f_0 and f_1 are defined in (9).

PROOF. The theorem follows from Theorem 6 and from Lemma 17. \square

Remark 19. Undoubtedly, the case when the polynomial is stable deserves special consideration. We will devote to it a section of Chapter II.

Remark 20. If we introduce the step function

$$U(\omega) \equiv \sum_{k: -\infty < \omega_k < \omega} \text{Ind}_{\omega_k}(R),$$

then (17) implies

$$\text{Ind}_a^b(R) = \int_a^b dU(\omega) = U(b-0) - U(a+0). \quad (18)$$

If we could calculate values of the function $U(\omega)$ without calculating the roots ω_k , we could conveniently apply Theorem 18.

Problem 21. *What is the Cauchy index of the logarithmic derivative*

$$R \equiv \frac{d}{d\omega} \ln f = \frac{f'}{f}.$$

of a real polynomial f ?

Hint: It equals the number of *distinct* roots of the polynomial f in the interval (a, b) .

Problem 22. *The choice of the imaginary axis for computing values of Δ_p is most convenient. But we could take any other line that goes through 0, except for the real axis. Prove this and explain the problem with the real axis.*

Problem 23. *Let us draw a graph of the rational function R on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ instead of the plane $\mathbb{R}^1 \times \mathbb{R}^1$. This is useful since one of the points on the circle \mathbb{S}^1 must correspond to the point at infinity on the axis \mathbb{R}^1 . Verify that such a «graph» is a closed curve on the torus. Which geometric (more exactly, topological) meaning does $\text{Ind}_{-\infty}^{+\infty}(R)$ acquire?*

THEOREM 18 IS A BETTER TOOL THAN THEOREM 6. HOWEVER, WE SEEM TO GO AROUND IN CIRCLES: TRYING TO AVOID AN EXPLICIT CALCULATION OF THE ROOTS OF THE INITIAL POLYNOMIAL p , WE CAME TO THE NECESSITY OF CALCULATING THE ROOTS OF ANOTHER POLYNOMIAL f_0 ! THE NEXT SECTION WILL SHOW THAT THIS IS IN FACT UNNECESSARY.

4. Sturm method

In the book [19], one can find an elegant theorem of Sturm on counting the number of real roots of a polynomial in a given interval. In fact, this theorem — or, more exactly, its method — has a wider scope.

A finite sequence of polynomials $\{f_0, f_1, \dots, f_n\}$ is called a *Sturm sequence* on the interval (a, b) if

$$f_0(c) = 0 \quad (a < c < b) \quad \Rightarrow \quad f_1(c) \neq 0, \quad (19)$$

$$f_n(c) \neq 0, \quad \forall c \in (a, b), \quad (20)$$

$$f_k(c) = 0 \quad (0 < k < n, \quad a < c < b) \quad \Rightarrow \quad f_{k-1}(c) f_{k+1}(c) < 0. \quad (21)$$

Given a Sturm sequence, let us introduce the integer-valued function $V(x)$ defined to be the number of sign changes in the sequence $\{f_0(x), f_1(x), \dots, f_n(x)\}$. The domain of this function is the interval (a, b) from which the roots of the polynomials in the sequence are excluded. At these points, the function $V(x)$ can have a discontinuity of the first kind. However, $V(x)$ does not have too many discontinuities:

Lemma 24. *If $c \in (a, b)$ is not a zero of odd multiplicity of the initial polynomial f_0 , then $V(c+0) = V(c-0)$.*

PROOF. If $f_k(c) = 0$, then $k < n$ in accordance with (20). Now let $k > 0$; then (21) implies that $f_{k-1}(c)$ and $f_{k+1}(c)$ are nonzero and have different signs. Therefore, the subsequence $\{f_{k-1}(c), f_k(c), f_{k+1}(c)\}$ ($0 < |x - c| < \varepsilon$) contains exactly one sign change regardless of the sign of $f_k(x)$ for x in a small neighborhood of c .

But if c is a zero of the polynomial f_0 of even multiplicity, then $f_0(x)$ does not change sign in the punctured neighbourhood $0 < |x - c| < \varepsilon$. By (19), this also applies to $f_1(x)$. Thus, the subsequence $\{f_0(x), f_1(x)\}$ has the same number of sign changes (0 or 1) to the left of the point c as it does to the right of c . \square

Lemma 25. *If $c \in (a, b)$ is a zero of f_0 of odd multiplicity, then*

$$V(c+0) - V(c-0) = -\text{Ind}_c \left(\frac{f_1}{f_0} \right). \quad (22)$$

PROOF. If the index at c is equal to $+1$, then the function f_1/f_0 changes its sign from $-$ to $+$ when x goes through the point c . The subsequence $\{f_0(x), f_1(x)\}$ thus loses a sign change and $V(x)$ decreases by 1. In case the index is negative, the opposite holds. \square

Theorem 26 (Sturm).¹²

$$\text{Ind}_a^b \left(\frac{f_1}{f_0} \right) = V(a+0) - V(b-0). \quad (23)$$

PROOF. Use the two previous lemmata and the fact that the full increment of a step function is the sum of its increments at points of discontinuity.

$$V(b-0) - V(a+0) = \sum [V(c+0) - V(c-0)],$$

where summation is taken over all discontinuities of the function V , that is, over all zeros of f_0 of odd multiplicity that lie in (a, b) . \square

¹²We foresaw the existence of such a formula — see Remark 20. It would resemble (18) even more if we defined $V(x)$ as the number of *sign retentions* rather than the number of sign changes.

Remark 27. Note that neither in the definition of Sturm sequences nor in the proof of the Sturm theorem was it essential that f_0, \dots, f_n are polynomials. Only the fact that these functions are continuous and have finitely many roots was used¹³.

Let us consider an important special case of Theorem 26.

A Sturm sequence $\{f_0, f_1, \dots, f_n\}$ is called *regular* if

$$\deg f_k = n - k \quad (k = 0, 1, \dots, n).$$

Theorem 28. *Let h_k be the leading coefficient of a polynomial f_k from a regular Sturm sequence $\{f_0, f_1, \dots, f_n\}$. Then*

$$\text{Ind}_{-\infty}^{+\infty} \left(\frac{f_1}{f_0} \right) = n - 2v(h_0, h_1, \dots, h_n), \quad (24)$$

where $v(h_0, h_1, \dots, h_n)$ is the number of sign changes in the sequence $\{h_0, h_1, \dots, h_n\}$.

PROOF. For large $|x|$, the sign of $f_k(x)$ coincides with the sign of its leading term $h_k x^{n-k}$. Therefore,

$$V(+\infty) = v(h_0, h_1, \dots, h_n),$$

$$V(-\infty) = v(h_n, -h_{n-1}, \dots, (-1)^n h_0) = n - v(h_0, h_1, \dots, h_n)$$

(the latter quantity is the number of *sign retentions*, which, together with the number of sign changes in the sequence $\{h_0, \dots, h_n\}$, sums up to n). \square

It remains to discuss how a Sturm sequence can be constructed from an initial pair of polynomials $\{f_0, f_1\}$. For the case $\deg f_0 > \deg f_1$, which we focus on, a modified Euclidean algorithm can be used. The modification consists in changing the sign of the remainder at each step:

$$f_{k-1} = d_k f_k - f_{k+1}, \quad \deg f_{k+1} < \deg f_k. \quad (25)$$

It does not change the original meaning of the algorithm since, at the last step, we will still obtain $\gcd(f_0, f_1)$, but it guarantees that the condition (21) is satisfied. Regarding the conditions (19) and (20), for this construction they are equivalent (prove it).

Remark 29. This is not the only possible method for constructing a Sturm sequence.

Remark 30. «Though the Sturm method is excellent in theory, it is not convenient in practice due to the enormous number of numerical coefficients of various powers of x in a sequence of Sturm functions when an equation of high enough degree is given.» (P. L. Chebyshev)

Problem 31. *How can we construct a Sturm sequence if $\deg f_0 \leq \deg f_1$?*

Problem 32. *Use the result of Problem 21 and suggest an algorithm for counting the number of distinct roots of a polynomial f on an interval (a, b) . Use this algorithm to count the number of roots of the polynomial p on the imaginary axis.*

WE HAVE SOLVED THE ROUTH-HURWITZ PROBLEM, AND NOT ONLY FOR REAL BUT ALSO FOR COMPLEX POLYNOMIALS. WE FOUND OUT THAT THIS PROBLEM IS ALGORITHMICALLY EQUIVALENT TO FINDING THE GREATEST COMMON DIVISOR OF TWO POLYNOMIALS AND IS THEREFORE RATHER SIMPLE. IT TURNS OUT THAT, WHEN USED TO EXAMINE THE STABILITY OF REAL POLYNOMIALS, IT SIMPLIFIES FURTHER DUE TO THE SPECIFIC STRUCTURE OF THE INITIAL POLYNOMIALS (11). THIS WILL BE TAKEN UP IN THE NEXT SECTION.

¹³Thus, the Sturm method solves problems of *algebra* by means of *analysis*. As a result, it is disliked by both algebraists and analysts.

5. Routh scheme

Let polynomials f_0 and f_1 be defined as in (11). If $a_1 \neq 0$, then the quotient and the remainder in (25) are

$$d_1(\omega) = c\omega \quad \left(c = \frac{a_0}{a_1}\right), \quad (26)$$

$$f_2(\omega) = (a_2 - ca_3)\omega^{n-2} - (a_4 - ca_5)\omega^{n-4} + \dots. \quad (27)$$

We see that, first of all, f_2 has the same structure as f_0 and f_1 , and if its leading coefficient is nonzero, then the same procedure can be applied to the pair $\{f_1, f_2\}$. Secondly, the coefficients of f_2 occur in the second row of the rectangular matrix

$$\begin{pmatrix} a_1 & a_3 & a_5 & \dots \\ a_0 & a_2 & a_4 & \dots \end{pmatrix} \quad (28)$$

after the Gaussian elimination of the entry a_0 .

This is the basis for the computational *Routh scheme*. In textbooks and handbooks, the Routh scheme is usually described in a form suitable for computations by hand. Thanks to the progress of programming, mathematics now has new tools for recording its algorithms. Let us use standard Pascal.

Let the coefficients of the polynomial (8) be stored in the array `var h: array[0..n] of real`; The transition from $\{f_0, f_1\}$ to $\{f_1, f_2\}$ described in (26)-(27) corresponds to the formal transition from the polynomial p to the polynomial

$$a_1 z^{n-1} + (a_2 - ca_3) z^{n-3} + a_3 z^{n-3} + (a_4 - ca_5) z^{n-4} + \dots,$$

whereupon the process goes on provided that corresponding coefficients are not zero. This algorithm is realized by the routine `Routh`, which returns the logical value `true` after a normal completion; in this case, it places the leading coefficients h_0, h_1, \dots, h_n of the Sturm sequence polynomials into the array `h`, to which Theorem 4.2 is then applied.

```
function Routh(var h:array[0..n] of real):boolean;
var k,j:integer;
    c:real;
begin
    k:=1;
    while (k<n-1) and (h[k]<>0) do
        begin
            c:=h[k-1]/h[k];
            k:=k+1;
            j:=k;
            repeat
                h[j]:=h[j]-c*h[j+1];
                j:=j+2
            until j>=n
        end;
    Routh:=(k=n-1)
end {Routh};
```

After the k th step, «the tail» `[k..n]` of the array `h` is filled with the alternating coefficients of the polynomials $\{f_k, f_{k+1}\}$, but «the head» contains the leading coefficients of all preceding polynomials of the sequence as some sort of «useful trash».

We begin our analysis of the algorithm by criticizing it. If `Routh(h)=false`, then we will only know that the given polynomial generates a nonregular sequence, but the

question of where its roots are located will remain open. This is not a dead end since the original Sturm method can work with these nonregular situations too. We will not go into details, since all is well in the case of interest to us:

Theorem 33. *A polynomial is stable if and only if $\text{Routh}(\mathbf{h})=\text{true}$ and h_0, h_1, \dots, h_n are not zero and of the same sign.*

PROOF. By Theorems 18 and 26, stability implies $V(-\infty) - V(+\infty) = n$. On the other hand, since $V(-\infty) \leq n$ and $V(+\infty) \geq 0$, this must be the extreme case $V(-\infty) = n$, $V(+\infty) = 0$. The former equality shows that the length of a Sturm sequence is maximal and equal to $n+1$; such a sequence is regular. The latter equality implies that all h_k are of the same sign. Hereby the necessity is proved. Now, Theorems 28 and 18 imply the sufficiency. \square

Here comes a pleasant surprise of the algorithm¹⁴. As **Routh** was developed, it was assumed that the polynomial has no roots on the imaginary axis and that it generates a regular Sturm sequence

$$f_k(z) = h_k z^{n-k} + \dots \quad (h_k \neq 0, \quad k = 0, 1, \dots, n).$$

But **Routh** does not require division by the last two coefficients h_n and h_{n-1} , so these coefficients, unlike the rest, can take the value zero. Thus, the scope of **Routh** is wider than originally intended. This property of the algorithm comes in handy. The point is that the loss of stability of a fixed point of the system (1)¹⁵ is accompanied by effects determined by precisely how the eigenvalues of the matrix A leave the left half-plane¹⁶. The following two scenarios are most common:

- a simple real eigenvalue crosses the imaginary axis at the point 0 when it enters the right half-plane;
- a pair of simple non-real eigenvalues crosses the imaginary axis at points $\pm i\omega$ ($\omega \neq 0$).

The Routh algorithm can distinguish between these two variants:

Theorem 34. *Let $\text{Routh}(\mathbf{h})=\text{true}$. Then,*

- a) *if $h_{n-1} \neq 0$, $h_n \neq 0$, then the polynomial has no roots on the imaginary axis, and*

$$n_+ = v(h_0, \dots, h_n), \quad n_- = n - v(h_0, \dots, h_n); \quad (29)$$

- b) *if $h_{n-1} \neq 0$, $h_n = 0$, then the polynomial has one simple root on the imaginary axis at the point 0, and*

$$n_+ = v(h_0, \dots, h_{n-1}), \quad n_- = n - 1 - v(h_0, \dots, h_{n-1}); \quad (30)$$

- c) *if $h_{n-1} = 0$, $h_{n-2}h_n < 0$, then the polynomial has no roots on the imaginary axis, and*

$$n_+ = v(h_0, \dots, h_{n-2}) + 1, \quad n_- = n - v(h_0, \dots, h_{n-2}) - 1; \quad (31)$$

¹⁴Why keep talking about sad things?

«... לאמיר בעסער רעדן פֿון עפעס פֿרייעכערס : וואָס הערט זיך עפעס מכה דער חלירע אין אָדעס? ...»

[Let us talk of something amusing. What is the news about cholera in Odessa?]» (Sholom Aleichem)

¹⁵Under perturbation of the system (1) and therefore of A [translators' remark].

¹⁶A new fixed point or a *limit cycle* can branch from a fixed point that loses stability. *Bifurcation theory* studies such phenomena.

- d) if $h_{n-1} = 0$, $h_{n-2}h_n > 0$, then the polynomial has two simple roots on the imaginary axis at the points $\pm i\omega$ ($\omega \neq 0$), and

$$n_+ = v(h_0, \dots, h_{n-2}), \quad n_- = n - 2 - v(h_0, \dots, h_{n-2}); \quad (32)$$

- e) if $h_{n-1} = 0$, $h_n = 0$, then the polynomial has one double root on the imaginary axis at the point 0, and

$$n_+ = v(h_0, \dots, h_{n-2}), \quad n_- = n - 2 - v(h_0, \dots, h_{n-2}). \quad (33)$$

PROOF. ¹⁷ The last three polynomials of the Sturm sequence are:

$$f_{n-2}(x) = h_{n-2}x^2 - h_n, \quad f_{n-1}(x) = h_{n-1}x, \quad f_n(x) = h_n.$$

In case (a), the sequence is regular. In the remaining cases, one or both polynomials f_{n-1} , f_n are identically zero. Recall that we deal with the Euclidean algorithm, therefore the construction of f_k must be stopped as soon as there is division without remainder in (25); then the last nonzero polynomial f_k is the greatest common divisor of the initial polynomials up to a numeric factor. We have:

$$\begin{aligned} d(\omega) &= \frac{1}{h_{n-1}} f_{n-1}(\omega) = \omega \quad \text{in case (b),} \\ d(\omega) &= \frac{1}{h_{n-2}} f_{n-2}(\omega) = \omega^2 - \frac{h_n}{h_{n-2}} \quad \text{in cases (c), (d), (e).} \end{aligned}$$

The statements of the theorem about the number and the location of roots of $p(z)$ on the imaginary axis follow from (12). It suffices to verify (29)–(33).

In the «safe» case (c), the last polynomial f_{n-2} is not equal to zero on the real axis, and the system $\{f_0, f_1, \dots, f_{n-2}\}$ is a Sturm sequence. Therefore we have to use Theorem 26 instead of Theorem 28.

Cases (b), (d) and (e) require correction of the initial polynomial $p(z)$. It should be divided by $i^l d(-iz)$, where $l = \deg d$, in order to remove its purely imaginary roots. At the same time, f_0 and f_1 should be divided by their greatest common divisor d . Their leading coefficients h_0 and h_1 do not change since we normalized d beforehand. Therefore the corrected polynomial requires no recalculations. We just have to take into account that its degree has decreased and to reuse Theorem 28. \square

Remark 35. A substantial difference between h_n and h_{n-1} is that the last component of the array does not get processed. It remains equal to the last term a_n of the polynomial p . On the contrary, the penultimate term of the array is subjected to the largest number of arithmetical operations.

Problem 36. Run **Routh** manually for polynomials of degree 3 and 4 and find necessary and sufficient conditions for the stability of these polynomials.

Problem 37. Program the routine

function IsStable(h:array[0..n] of real):boolean;

that inputs the array **h** of coefficients of a real polynomial p and returns **true** if the polynomial p is stable or **false** otherwise. Program also the routine **IsStableStodola** which differs from **IsStable** by testing a polynomial first for the Stodola condition. Is such an improvement of **IsStable** useful?

Problem 38. The following source code is written in APL (see [12]):

¹⁷We recommend that the reader give a proof on his/her own.

```

▽ B ← WhatIsIt A
  LOOP: ((1=1↓ ρA) ∨ (A[2;1] ≤ 0)) /EXIT
  A ← (2 2 ρ 0 1 1 -A[1;1] ÷ A[2;1]) + . × (0 1 ↓ A)
  → LOOP
  EXIT: B ← (A[2;1] > 0)
▽

```

The input parameter \mathbf{A} is a matrix of the form (28). Which algorithm does the function `WhatIsIt` realize? How can it be improved? Give a comparative analysis of Pascal versus APL as algorithmic languages.

SO WE NOW HAVE AT OUR DISPOSAL A SIMPLE (POSSIBLY UNIMPROVABLE) ALGORITHM FOR TESTING THE STABILITY OF A REAL POLYNOMIAL. WE JUST HAVE NO EXPLICIT FORMULÆ THAT EXPRESS THE OUTPUT OF THIS ALGORITHM. FOR COMPARISON, THE GAUSS ALGORITHM AND THE CRAMER FORMULÆ ARE COMPLEMENTARY IN THE THEORY OF LINEAR SYSTEMS: THE FORMER DESCRIBES HOW TO FIND A SOLUTION, THE LATTER HOW THIS SOLUTION LOOKS.

6. Hurwitz theorem

Given a polynomial (8) with real coefficients, consider a corresponding $n \times n$ matrix of the following structure:

$$\mathcal{H}_p \equiv \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \cdots \\ a_0 & a_2 & a_4 & a_6 & \cdots \\ 0 & a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 & a_4 & \cdots \\ 0 & 0 & a_1 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (34)$$

(the coefficients a_0, \dots, a_n are not enough to fill the rows, but we set $a_{n+1} = a_{n+2} = \dots = 0$). The matrix \mathcal{H}_p is called the *Hurwitz matrix* of the polynomial p . Let us denote by η_k the leading principal minor of this matrix formed from the first k rows and columns. It is easy to check that there is only one (diagonal) nonzero term in the last column of \mathcal{H}_p . It equals a_n . Therefore

$$\eta_n = \eta_{n-1} a_n. \quad (35)$$

Lemma 39. *In the regular case,*

$$h_1 = \eta_1, \quad h_2 = \frac{\eta_2}{\eta_1}, \quad \dots, \quad h_n = \frac{\eta_n}{\eta_{n-1}}. \quad (36)$$

PROOF. First note that the matrix \mathcal{H}_p consists of blocks of type (28), which are in turn made up from the coefficients of the polynomials f_0 and f_1 . Let us reduce the matrix \mathcal{H}_p to upper triangular form using Gaussian elimination without pivoting. By (26)-(27), the elimination of the term a_0 from the second, fourth etc. rows leaves these rows filled with the coefficients of f_2 , the next polynomial of the sequence. The first row and column are no longer needed. Crossing them out (temporarily), we obtain an $(n-1) \times (n-1)$ matrix of the same «Hurwitz» structure formed from the coefficients of the polynomials f_1 and f_2 . Repeating the same procedure, we will arrive at a triangular matrix

$$\begin{pmatrix} h_1 & * & * & \cdots & * \\ 0 & h_2 & * & \cdots & * \\ 0 & 0 & h_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_n \end{pmatrix}$$

whose k th row consists of the coefficients of the polynomial f_k , starting with the leading coefficient h_k on the main diagonal.

Each elementary transformation that we applied consisted in subtracting the *preceding* row (multiplied by a suitable number) from a given row. Such transformations preserve not only the determinant $|\mathcal{H}_p| = \eta_n$ but also all leading principal minors η_k . Consequently,

$$\eta_k = h_1 h_2 \cdots h_k \quad (k = 1, 2, \dots, n), \quad (37)$$

which implies (36). \square

Theorem 40 (Hurwitz). *A polynomial*

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0, a_1, \dots, a_n \in \mathbb{R}; a_0 > 0) \quad (38)$$

is stable if and only if all leading principal minors of its Hurwitz matrix \mathcal{H}_p are positive:

$$\eta_k > 0 \quad (k = 1, 2, \dots, n). \quad (39)$$

PROOF. The assertion of the theorem follows immediately from the preceding Lemma and Theorems 33 and 34. \square

Remark 41. Let us assume that the polynomial p depends continuously on one or several parameters and is initially stable. Lemma 6.1 and Theorem 5.2 imply that its stability is maintained for as long as $|\mathcal{H}_p| = \eta_n$ is not zero¹⁸. At the same time, owing to (34) and to Theorem 5.2 again, a sign change in a_n (for $\eta_{n-1} \neq 0$) corresponds to the transition of a simple real root from the left to the right half-plane. A sign change in η_{n-1} (for $a_n \neq 0$, $\eta_{n-2} \neq 0$) corresponds to the transition of a pair of simple complex conjugate roots from the left to the right half-plane.

Remark 42. The observation we just made appears useful: in order to «catch» the moment when a continuously varying polynomial loses stability, it is sufficient to check only the sign of the last minor η_n . This is, however, no reason to celebrate: Theorem 33 implies that, while the polynomial remains stable, the determinant $|\mathcal{H}_p|$ is best computed using the compact Gauss scheme, which yields all the leading principal minors η_k as a by-product anyway. In general, it makes no sense to apply the Hurwitz theorem for computations – to this end one should use the Routh scheme.

Remark 43. In 1914, A. Liénard and M. Chipart proposed a different criterion of polynomial stability. They established that a polynomial (38) of degree n *with positive coefficients* is stable if and only if the following conditions are satisfied:

$$\begin{aligned} \eta_2 > 0, \eta_4 > 0, \dots, \eta_{n-1} > 0, & \quad \text{if } n \text{ is odd,} \\ \eta_1 > 0, \eta_3 > 0, \dots, \eta_{n-1} > 0 & \quad \text{if } n \text{ is even.} \end{aligned} \quad (40)$$

The Liénard-Chipart conditions (40) look simpler than the Hurwitz conditions (39) since they contain half that many determinantal inequalities. Although the simplicity is misleading from the computational point of view (for reasons given in Remark 42), conditions (40) may be more useful for formal derivations, and the equivalence of (39) and (40) is very interesting from the theoretical point of view.

Problem 44. *Assuming the minors η_1, \dots, η_n are all nonzero, prove that the number of zeros of the polynomial p in the right (left) half-plane is equal to the number of sign variations (retentions) in the sequence $\left\{ a_0, \eta_1, \frac{\eta_2}{\eta_1}, \dots, \frac{\eta_n}{\eta_{n-1}} \right\}$.*

¹⁸Unlike the fish, the Hurwitz conditions «rot from the tail».

Problem 45. Let all η_k be nonnegative. Does this imply that all roots of p lie in the closed left half-plane?

Hint: No, it does not. Give a counterexample.

Problem 46. Prove that the statement of the Hurwitz theorem remains valid if a_0, a_1, a_2, \dots in (34) are the coefficients of the polynomial

$$p(z) \equiv a_0 + a_1 z + a_2 z^2 + \dots \quad (a_0, a_1, \dots \in \mathbb{R}; a_0 > 0). \quad (41)$$

Hint: if $p(0) \neq 0$, then the polynomials $p(z)$ and $q(z) \equiv z^n p(z^{-1})$ ($n = \deg p$) are simultaneously stable or unstable.

Problem 47. If $p(z)$ is an analytic function represented by the power series (41), then we can formally construct its infinite Hurwitz matrix and require the positivity of its leading principal minors. Give an example showing that the Hurwitz theorem does not generalize to analytic functions. In this connection, see [27, 1, 7].

OUR GOALS ARE ACHIEVED, AND WE ARE DONE WITH THE BASIC MATERIAL. IN CHAPTER II WE WILL CONSIDER MANY VARIATIONS ON THE TOPIC OF STABLE POLYNOMIALS. THAT OPTIONAL MATERIAL CAN BE USED FOR SEMINARS AS WELL AS FOR INDEPENDENT STUDY.

Appendix to Chapter I

Here we consider the stability problem of the Lorentz system posed in the Introduction¹⁹:

$$\dot{X} = \sigma Y - \sigma X, \quad \dot{Y} = rX - Y - XZ, \quad \dot{Z} = XY - bZ. \quad (42)$$

1. Determining fixed points. A fixed point of the system (42) is a point $(X_k, Y_k, Z_k) \in \mathbb{R}^3$ that annihilates the right hand sides in (42), i.e., a solution to the following system of algebraic equations:

$$\sigma Y_k - \sigma X_k = 0, \quad rX_k - Y_k - X_k Z_k = 0, \quad X_k Y_k - bZ_k = 0. \quad (43)$$

One of the solutions to (43) is easy to spot: it is the zero fixed point

$$(X_0, Y_0, Z_0) = (0, 0, 0). \quad (44)$$

In addition, the system (43) has two more solutions

$$(X_{1,2}, Y_{1,2}, Z_{1,2}) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \quad (45)$$

Remark 48. The zero fixed point exists for all positive values of the parameters σ, b, r . A pair of nonzero fixed points (45) bifurcates from it when $r > 1$.

2. Linearization. In order to linearize the system $\dot{x} = f(x)$ in the neighborhood of a fixed point x_k (where $f(x_k) = 0$), one has to replace the function $f(x)$ by its linear form $A_k(x - x_k)$ where $A_k = f'(x_k)$ is the Jacobian matrix consisting of the partial derivatives of the function $f(x)$ taken at the point x_k . In our case,

$$A_k = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_k & -1 & -X_k \\ Y_k & X_k & -b \end{pmatrix} \quad (k = 0, 1, 2).$$

¹⁹We remind the reader that the parameters b, r, σ are assumed to be positive [translators' remark].

3. Computing the characteristic polynomial. We have

$$p_k(\lambda) = |\lambda I - A_k| = (\lambda + \sigma)(\lambda + 1)(\lambda + b) + \sigma X_k^2 + \sigma(Z_k - r)(\lambda + b) + X_k^2(\lambda + \sigma)$$

since $X_k = Y_k$ for all three fixed points.

For the zero fixed point (44), we obtain

$$p_0(\lambda) = (\lambda + b)[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)];$$

for the fixed points (45), we get

$$p_{1,2}(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \quad \text{where} \\ a_0 = 1, \quad a_1 = \sigma + b + 1, \quad a_2 = b(\sigma + r), \quad a_3 = 2\sigma b(r - 1).$$

4. Stability of fixed points. The Hurwitz theorem is in fact not needed to analyze the stability of the polynomial p_0 – the elementary Vieta's Theorem is enough. This polynomial is stable for $0 < r < 1$.

Remark 49. At $r = 1$, one of the roots of the polynomial p_0 crosses the imaginary axis and enters the right half-plane. The zero fixed point loses its stability, and exactly then the fixed points (45) bifurcate from it.

For the polynomials $p_{1,2}$ of degree 3, the Hurwitz matrix has the form

$$\begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix},$$

and the conditions of Theorem 40 say

$$a_1 > 0, \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} a_3 > 0.$$

As $a_1 > 0$, $a_3 > 0$ for $r > 1$, there remains just one inequality

$$a_1 a_2 - a_0 a_3 = b(\sigma + b + 1)(\sigma + r) - 2b\sigma(r - 1) > 0,$$

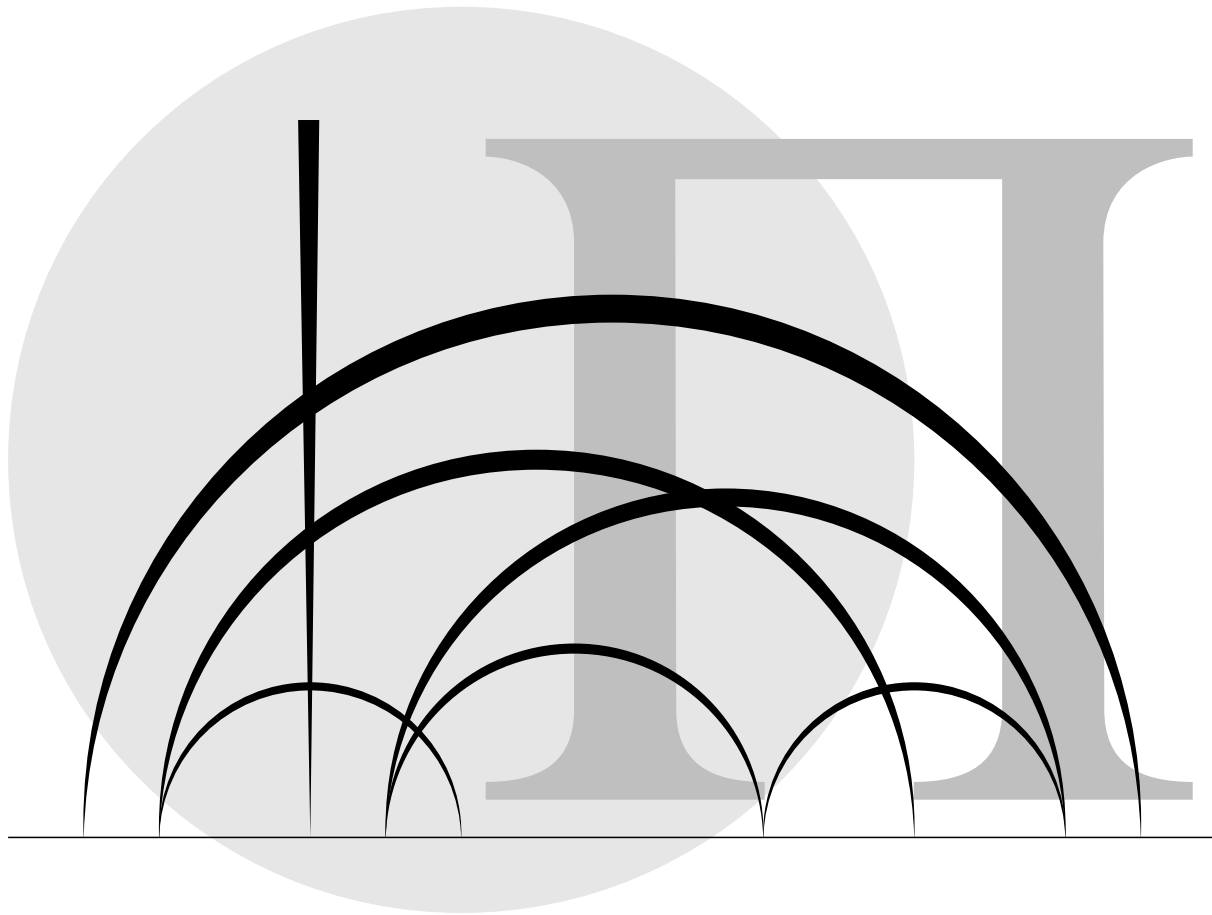
which can be viewed as an answer. However, it is better to represent the answer as follows:

$$1 < r < r_* \quad \text{where} \quad r_* = \begin{cases} \sigma \frac{\sigma+b+3}{\sigma-b-1} & \sigma > b + 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 50. The fixed points that bifurcate from $(0, 0, 0)$ are initially stable. When the parameter r reaches its critical value r_* , they lose their stability since, by Theorem 34 (also see Remark 41), a pair of nonreal roots of the polynomial $p_{1,2}$ crosses the imaginary axis. One can in fact show that, at $r = r_*$, *limit cycles* branch from the fixed points that lose stability. Further discussion of these difficult and interesting questions is beyond the scope of our notes; these questions are subject of bifurcation theory.

CHAPTER 2

Extensions



"Voilà le sujet simplifié, argumentum omni denudatum ornamento. Je ferais avec cela, continua le jésuite, deux volumes de la taille de celui-ci."

Et, dans son enthousiasme, il frappait sur le saint Chrysostome in-folio qui faisait plier la table sous son poids.

D'Artagnan frémit.

Alexander Dumas. LES TROIS MOUSQUETAIRES.
Chapitre XXVI. La thèse d'Aramis.

7. Stieltjes Fractions

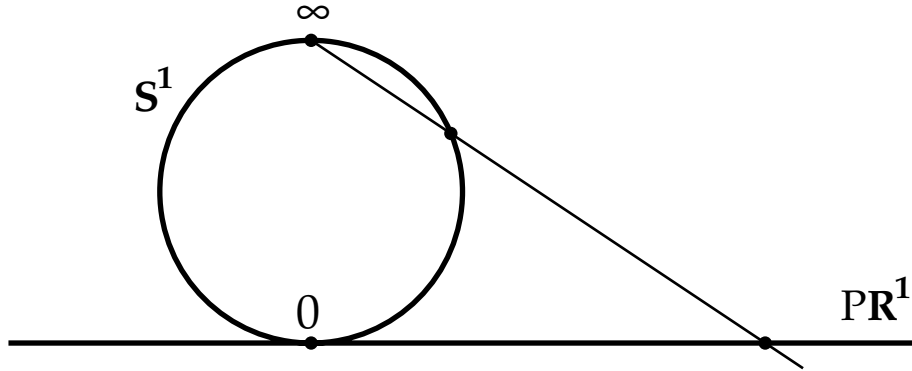
The definition of the Cauchy index $\text{Ind}_{-\infty}^{+\infty}(R)$ given in § 3 presupposed that the rational function R vanishes at infinity. Let us try to generalize this notion to arbitrary rational functions. A useful idea is contained in Problem 23: the point at infinity ∞ should be «made equal» to the other points, and it makes sense to consider the function R as a map of the *projective line*¹ $\mathbb{P}\mathbb{R}^1 \equiv \mathbb{R}^1 \cup \{\infty\}$ into itself. In particular, if $R = f_1/f_0$ and $\deg f_1 > \deg f_0$, then we will view R as having a pole at ∞ of order $\deg f_1 - \deg f_0$. If that pole is of odd order, then, analogously to (16) (§ 3), we let

$$\text{Ind}_{\infty}(R) = \begin{cases} +1, & \text{if } R(+\infty) < 0 < R(-\infty) \\ -1, & \text{if } R(+\infty) > 0 > R(-\infty), \end{cases} \quad (46)$$

in all other cases, let $\text{Ind}_{\infty}(R) = 0$. This summand should be added to (17) (§ 3):

$$\text{Ind}_{\mathbb{P}\mathbb{R}}(R) \equiv \text{Ind}_{-\infty}^{+\infty}(R) + \text{Ind}_{\infty}(R). \quad (47)$$

Remark 51. Incidentally, (46) implies that rays $(C, +\infty)$ should be viewed as *left* half-neighborhoods of the point ∞ , and rays $(-\infty, -C)$ as its *right* neighborhoods. The projective line is in one-to-one correspondence with the circle \mathbb{S}^1 , as illustrated below:



Remark 52. Let $d(\omega) = c\omega^\nu + \dots$ ($\nu = \deg d$) be a polynomial. Then

$$\text{Ind}_{\infty}(d) = \begin{cases} -\text{sign } c, & \text{if } \nu \text{ is odd,} \\ 0, & \text{if } \nu \text{ is even.} \end{cases} \quad (48)$$

Now note how the index $\text{Ind}_{\mathbb{P}\mathbb{R}}(R)$ changes when various «projective» maps act on R .

Lemma 53. *If $d \in \mathbb{R}$ is a constant, then $\text{Ind}_{\mathbb{P}\mathbb{R}}(d + R) = \text{Ind}_{\mathbb{P}\mathbb{R}}(R)$.*

PROOF. The addition of a constant does not change the behaviour of a function near its poles. \square

Lemma 54. *If d is a polynomial and $R(\infty) = 0$, then $\text{Ind}_{\mathbb{P}\mathbb{R}}(d + R) = \text{Ind}_{-\infty}^{+\infty}(R) + \text{Ind}_{\infty}(d)$.*

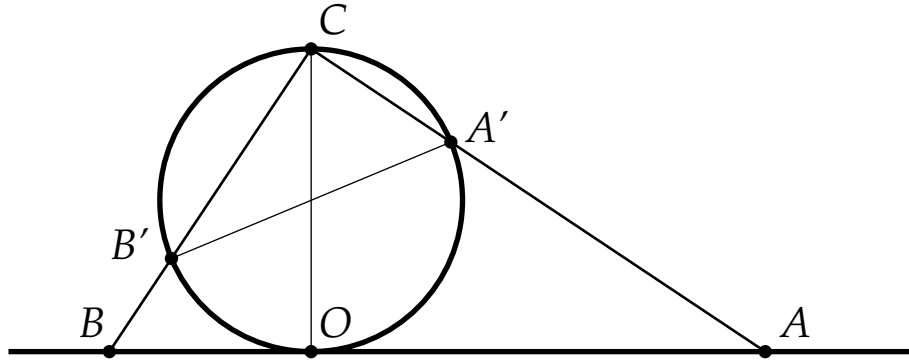
PROOF. See (46)–(47). \square

Lemma 55. $\text{Ind}_{\mathbb{P}\mathbb{R}}(-\frac{1}{R}) = \text{Ind}_{\mathbb{P}\mathbb{R}}(R)$.

¹For sure, the point is not just to add a new element to the set \mathbb{R}^1 ; we need to incorporate that element into the algebraic and topological structures that exist in \mathbb{R}^1 . More about projective spaces $\mathbb{P}\mathbb{R}^n$ can be found in [9, 13].

PROOF. The projective line $\mathbb{P}\mathbb{R}^1$ is divided by its *two* points 0 and ∞ into its positive \mathbb{R}^+ and negative \mathbb{R}^- rays. Let a variable ω traverse $\mathbb{P}\mathbb{R}^1$ and return to its starting point. Clearly, the number of crossings from \mathbb{R}^- to \mathbb{R}^+ must equal the number of reverse crossings. The crossings through ∞ occur at the poles of the function R , and they are accounted for in the sum (47) with the appropriate sign. The crossings through 0 occur at the zeros of R , i.e., at the poles of $\frac{1}{R}$, and they are accounted for in the analogous formula for $\text{Ind}_{\mathbb{P}\mathbb{R}}(\frac{1}{R})$. As a result, $\text{Ind}_{\mathbb{P}\mathbb{R}}(\frac{1}{R}) + \text{Ind}_{\mathbb{P}\mathbb{R}}(R) = 0$. \square

Remark 56. The transformations $\omega \mapsto \omega + d$ are called *shifts* of the projective line $\mathbb{P}\mathbb{R}^1$. The point $-\frac{1}{\omega}$ is referred to as the *polar* of $\omega \in \mathbb{P}\mathbb{R}^1$. If the diameter of the circle on Fig. 2 is equal to 1, then the polar B of the point A is constructed by drawing the perpendicular BC to AC . By a theorem from elementary geometry, $OA \cdot OB = OC^2$. Another theorem (on a subtended and a central angle) implies that $A'B'$ is a diameter.



Alternating between shifts $\omega \mapsto \omega + d$ and the polar transformation $\omega \mapsto -\frac{1}{\omega}$, we obtain a *continued fraction*. The right-hand sides of the following formula are examples of continued fractions:

Lemma 57. *If $\alpha\delta - \beta\gamma = 1$, then*

$$\frac{\alpha\omega + \beta}{\gamma\omega + \delta} = \begin{cases} \frac{1}{\gamma} - \frac{1}{\gamma - \frac{1}{\frac{1}{\gamma} - \frac{1}{\alpha\gamma - \frac{1}{\frac{\delta}{\gamma} + \omega}}}} & (\gamma \neq 0), \\ (\alpha\beta + \alpha) - \frac{1}{\frac{1}{\alpha} - \frac{1}{\alpha - \frac{1}{\omega}}} & (\gamma = 0). \end{cases}$$

Lemma 58. *If $\alpha\delta - \beta\gamma = 1$, then*

$$\frac{\alpha\omega + \beta}{\gamma\omega + \delta} = \begin{cases} \frac{1}{\gamma - \frac{1}{\gamma - \frac{1}{\gamma - \frac{1}{\gamma - \frac{1}{\alpha\gamma - \frac{\delta}{\frac{\delta}{\gamma} + \omega}}}}}} & (\gamma \neq 0), \\ (\alpha\beta + \alpha) - \frac{1}{\frac{1}{\alpha - \frac{1}{\alpha - \frac{1}{\omega}}}} & (\gamma = 0). \end{cases} \quad (49)$$

PROOF. By straightforward calculation. \square

The fractional-linear map

$$\Phi : \omega \mapsto \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \quad (\alpha, \beta, \gamma, \delta \in \mathbb{R}; \alpha\delta - \beta\gamma = 1)$$

performs a *projective transformation* $\mathbb{P}\mathbb{R}^1 \rightarrow \mathbb{P}\mathbb{R}^1$. Such transformations form the *group*² denoted by **SL**(2, \mathbb{R}).

Theorem 59. *The Cauchy index is invariant under the action of the group **SL**(2, \mathbb{R}) on rational functions:*

$$\text{Ind}_{\mathbb{P}\mathbb{R}}(\Phi \circ R) = \text{Ind}_{\mathbb{P}\mathbb{R}}(R), \quad \forall \Phi \in \mathbf{SL}(2, \mathbb{R}).$$

PROOF. Apply Lemmata 57, 53, 55. \square

Now let us explore connections between projective geometry, continued fractions, and the Euclidean algorithm. For definiteness, suppose that $R(\infty) = 0$, so that $R = f_1/f_0$, $\deg f_1 < \deg f_0$ in Sections 3 and 4. Run the modified Euclidean algorithm, which we used in § 4 to construct Sturm sequences:

$$f_{k-1} = d_k f_k - f_{k+1} \quad (k = 1, \dots, m; \deg f_{k+1} < \deg f_k; f_{m+1} = 0). \quad (50)$$

If

$$R_k \equiv \frac{f_{k+1}}{f_k} \quad (k = 0, 1, \dots, m; R_0 = R, R_m = 0), \quad (51)$$

then (50) yields the recurrence relation

$$-R_{k-1} = -\frac{1}{d_k - R_k}, \quad (52)$$

²One can find information about this group – and other things – in [3, §5].

and we obtain the following expansion of the rational function $-R$ into a continued fraction:

$$-R = -\frac{1}{d_1 - \frac{1}{d_2 - \frac{1}{\ddots - \frac{1}{d_m}}}}, \quad (53)$$

A functional continued fraction of type (53), where d_1, \dots, d_m are polynomials, is called a Stieltjes continued fraction (see [11, 28, 29, 30, 31]).

Theorem 60. *If a rational function R is represented by a continued fraction (53), then*

$$\text{Ind}_{\mathbb{P}\mathbb{R}}(R) = -\sum_{k=1}^m \text{Ind}_{\infty}(d_k). \quad (54)$$

PROOF. Apply Lemmata 54 and 55 inductively, using the relation (52). \square

Remark 61. Theorem 60 shows a way to compute Cauchy indices, which parallels Schur's method. It uses the same (Euclidean) algorithm, but its validity is established by different reasoning.

Now consider the extreme case when the function R is generated by a stable polynomial, as described in § 3. It is time to give such functions a name. We will call a rational function $R = f_1/f_0$ *proper* if $\deg f_1 < \deg f_0$ and $\text{Ind}_{\mathbb{P}\mathbb{R}}(R) = \deg f_0$; the class of all proper functions will be denoted by \mathcal{R} .

Theorem 62. *$R \in \mathcal{R}$ if and only if*

$$-R(\omega) = -\frac{1}{\alpha_1\omega + \beta_1 - \frac{1}{\alpha_2\omega + \beta_2 - \frac{1}{\ddots - \frac{1}{\alpha_n\omega + \beta_n}}}}, \quad (55)$$

where $\beta_1, \dots, \beta_n \in \mathbb{R}$, $\alpha_1, \dots, \alpha_n > 0$.

PROOF. Formula (50) implies that $n = \sum_{k=1}^m \deg d_k$, whereas (48) implies the inequality $-\text{Ind}_{\infty}(d) \leq \deg d$, where equality occurs if and only if $d(\omega) = \alpha\omega + \beta$ ($\beta \in \mathbb{R}$, $\alpha > 0$). A comparison with (54) now shows that the polynomials d_1, \dots, d_m in (53) must satisfy exactly these conditions, and their number must be exactly n . \square

Remark 63. The transformations $z \mapsto \alpha z + \beta$ ($\beta \in \mathbb{R}$, $\alpha > 0$) and $z \mapsto -\frac{1}{z}$ map the complex *upper* half-plane $\{z \in \mathbb{C} : \text{Im } z > 0\}$ into itself. Therefore, this property is inherited by the function $-R$ if it admits an expansion of type (55). This theme will be taken up again in § 8, but our next problem already has a hint of a variation on this idea:

Problem 64. *Let*

$$R(\omega) = \frac{1}{\alpha_1\omega + \beta_1 + \frac{1}{\alpha_2\omega + \beta_2 + \frac{1}{\ddots + \frac{1}{\alpha_n\omega + \beta_n}}}}, \quad (56)$$

where $\alpha_k, \beta_k > 0$ ($k = 1, 2, \dots, n$). Then the function R maps the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ into itself; all its poles and zeros must lie in the left half-plane.

Problem 65. Consider a tridiagonal matrix

$$A = \begin{pmatrix} a_1 & c_2 & 0 & \dots & 0 & 0 \\ b_2 & a_2 & c_3 & \dots & 0 & 0 \\ 0 & b_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & c_n \\ 0 & 0 & 0 & \dots & b_n & a_n \end{pmatrix}.$$

Prove that

- all eigenvalues of A are real and simple whenever $b_2c_2, \dots, b_nc_n > 0$ and $a_1, \dots, a_n \in \mathbb{R}$;
- all eigenvalues of A lie in the open right (resp., left) half-plane whenever $b_2c_2, \dots, b_nc_n < 0$ and $a_1, \dots, a_n > 0$ (resp., < 0).

Hint: Consider the rational function $R(\lambda) \equiv \Delta_{n-1}(\lambda)/\Delta_n(\lambda)$ where $\Delta_n(\lambda)$ denotes the determinant of the matrix $\lambda I - A$ and $\Delta_{n-1}(\lambda)$ denotes its principal minor obtained by omitting its last row and column. For this function, obtain a decomposition of type (55) or (56) and use the idea from Remark 63 and Problem 64.

Problem 66. Let $A : \mathfrak{B} \rightarrow \mathfrak{B}$ be a bounded linear operator on a Banach space \mathfrak{B} , let $u \in \mathfrak{B}$ be a vector in \mathfrak{B} , and let $\varphi \in \mathfrak{B}'$ be a bounded linear functional acting on \mathfrak{B} such that $\varphi(u) \neq 0$. We introduce

- the subspace $\mathfrak{B}_1 = \{x \in \mathfrak{B} : \varphi(x) = 0\} \subset \mathfrak{B}$,
- the operator $A_1 : \mathfrak{B}_1 \rightarrow \mathfrak{B}_1$, $A_1 : x \mapsto Ax - \frac{\varphi(Ax)}{\varphi(u)} u$,
- the vector $u_1 \in \mathfrak{B}_1$ by $u_1 = Au - \frac{\varphi(Au)}{\varphi(u)} u$,
- the functional $\varphi_1 \in \mathfrak{B}'_1$, $\varphi_1 : x \mapsto \varphi(Ax)$,

and consider the two analytic functions

$$R(\lambda) \equiv \varphi((\lambda I - A)^{-1}u), \quad R_1(\lambda) \equiv \varphi_1((\lambda I - A_1)^{-1}u_1).$$

Prove that

$$-R(\lambda) = -\frac{s_0^2}{\lambda s_0 - s_1 - R_1(\lambda)}, \quad \text{where } s_k \equiv \varphi(A^k u) \quad (k = 0, 1).$$

Consequently, if $\dim \mathfrak{B} < \infty$, then $\operatorname{Ind}_{-\infty}^{+\infty}(R) = \operatorname{sign} s_0 + \operatorname{Ind}_{-\infty}^{+\infty}(R_1)$.

Problem 67. Prove that the projective transformations

$$\Phi_\tau : \mathbb{P}\mathbb{R}^1 \rightarrow \mathbb{P}\mathbb{R}^1, \quad \Phi_\tau : \omega \mapsto \frac{\omega}{1 - \tau\omega} \quad (\tau \in \mathbb{R}) \quad (57)$$

form a one-parameter group, i.e.,

$$\Phi_t \circ \Phi_\tau = \Phi_{t+\tau}, \quad \Phi_\tau^{-1} = \Phi_{-\tau}$$

Problem 68. Using the notation of Problem 66, consider

$$A_t : \mathfrak{B} \rightarrow \mathfrak{B}, \quad A_t : x \mapsto Ax + t\varphi(x)u, \quad R_t(\lambda) \equiv \varphi((\lambda I - A_t)^{-1}u),$$

and let Φ_τ be defined as in (57). Prove that $R_{t+\tau} = \Phi_\tau \circ R_t$. Consequently, if $\dim \mathfrak{B} < \infty$, then

$$\operatorname{Ind}_{-\infty}^{+\infty}(R_t) = \operatorname{Ind}_{-\infty}^{+\infty}(R_\tau) \quad (\forall t, \tau \in \mathbb{R}).$$

Problem 69. Theorem 59 dealt with the left action of the group $\mathbf{SL}(2, \mathbb{R})$, where the group acts on the value of the function R . Prove that $\text{Ind}_{\mathbb{P}\mathbb{R}}(R)$ is also invariant under the right action of the group $\mathbf{SL}(2, \mathbb{R})$, where the group acts on the argument:

$$\text{Ind}_{\mathbb{P}\mathbb{R}}(R \circ \Phi) = \text{Ind}_{\mathbb{P}\mathbb{R}}(R), \quad \forall \Phi \in \mathbf{SL}(2, \mathbb{R}).$$

Problem 70. If polynomials f_0 and f_1 are as in § 5, then the function $R = f_1/f_0$ is odd, and Routh's algorithm, when it halts, yields an expansion of R into a continued fraction of the following form:

$$R(\omega) = \frac{1}{c_1\omega - \frac{1}{c_2\omega - \frac{1}{\ddots - \frac{1}{c_n\omega}}}}.$$

Prove that the following expansions are valid as well:

$$R(\omega) = \begin{cases} \omega^{-1} \cdot \frac{1}{c_1 - \frac{1}{c_2\omega^2 - \frac{1}{c_3 - \frac{1}{\ddots - \frac{1}{c_{2k-1}}}}}} & (n = 2k - 1), \\ \omega \cdot \frac{1}{c_1\omega^2 - \frac{1}{c_2 - \frac{1}{c_3\omega^2 - \frac{1}{\ddots - \frac{1}{c_{2k}}}}}} & (n = 2k). \end{cases}$$

Problem 71. The expression

$$\{R, z\} \equiv \frac{R'''(z)}{R'(z)} - \frac{3}{2} \left[\frac{R''(z)}{R'(z)} \right]^2$$

is called the differential Schwarz invariant, or the Schwarzian derivative of the function R (see [15]). Prove that

$$\{\Phi \circ R, z\} = \{R, z\} \quad \forall \Phi \in \mathbf{SL}(2, \mathbb{C}).$$

If $R(z) = s_0z + s_1z^2 + s_2z^3 + \dots$, prove that

$$\{R, 0\} = \frac{6}{s_0^2} \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}.$$

8. Hermite-Biehler Theorem

Let a **polynomial**

$$p(z) = a_0z^n + a_1z^{n-1} + \dots + a_n \quad (a_0 > 0; a_1, \dots, a_n \in \mathbb{R})$$

be **stable**. According to Theorem 3.1, the function $R \equiv f_1/f_0$, with real polynomials f_0 and f_1 that are defined by

$$i^{-n}p(i\omega) = f_0(\omega) - if_1(\omega) \quad (\omega \in \mathbb{R}), \quad (58)$$

is *proper*: $\text{Ind}_{-\infty}^{+\infty}(R) = \deg f_0 = n$. Since the sum (17) (or, equivalently, (47)) has no more than n terms ± 1 , this equality is possible only if the number of terms is exactly n , and they are all equal to $+1$. Therefore, the polynomial f_0 necessarily has n distinct real roots $\omega_1 < \dots < \omega_n$, and since $\deg f_0 = n$, it cannot have additional (nonreal or multiple) roots. Hence R splits into *elementary fractions* as follows:

$$R(z) = \sum_{k=1}^n \frac{\alpha_k}{z - \omega_k}, \quad \alpha_k = \text{Res}_{\omega_k}(R). \quad (59)$$

By (16), $\text{Ind}_{\omega_k}(R) = \text{sign } \alpha_k$, and all indices are equal to $+1$ in our case, so all residues α_k must be positive. Thus

$$\begin{aligned} \frac{\text{Im } R(z)}{\text{Im } z} &= - \sum_{k=1}^n \frac{\alpha_k}{|z - \omega_k|^2} < 0 \quad (\text{Im } z \neq 0), \\ \frac{dR(z)}{dz} &= - \sum_{k=1}^n \frac{\alpha_k}{(z - \omega_k)^2} < 0 \quad (\text{Im } z = 0). \end{aligned}$$

The function $-R$ therefore maps the upper half-plane $\{z: \text{Im } z > 0\}$ into itself, and monotonically increases between its consecutive (real) poles. So, between any two consecutive roots ω_{k-1}, ω_k ($k = 2, \dots, n$) of the denominator f_0 there must lie exactly one (simple) root of the numerator f_1 . Since $\deg f_1 \leq n - 1$, the polynomial f_1 cannot have any additional roots.

Furthermore, the formula (58) is *algebraic*, so it does not matter that the argument ω was initially assumed to be real. This formula may be re-written as

$$i^{-n}p(z) = f_0(-iz) - if_1(-iz) \quad (z \in \mathbb{C}). \quad (60)$$

Hence

$$p(z) = 0 \implies R(-iz) = -i \implies \frac{\text{Im}(-iz)}{\text{Im}(-i)} = \text{Re } z < 0.$$

The polynomial p is stable! We now summarize this walk «*there and back again*»:

Theorem 72. *Given a polynomial p , let polynomials f_0 and f_1 be defined by (58), and let $R \equiv f_1/f_0$. The following conditions are equivalent:*

- (1) *the polynomial p is stable;*
- (2) *the function R is proper;*
- (3) *the function R admits a representation of type (59), with $\omega_k \in \mathbb{R}$ and $\alpha_k > 0$ ($k = 1, \dots, n$);*
- (4) *the function $-R$ maps the upper half-plane into itself;*
- (5) *the roots of the polynomials f_0, f_1 are real and simple, between any two consecutive roots of f_0 there is exactly one root of f_1 , and³*

$$\exists \omega \in \mathbb{R} : f_1'(\omega) f_0(\omega) - f_0'(\omega) f_1(\omega) < 0. \quad (61)$$

Remark 73. The statement that conditions (1) and (5) are equivalent is called the *Hermite–Biehler* theorem. The two mathematicians obtained this result simultaneously (1879) and independently - see [11, 27, 18]; analogues for entire functions are given in [7].

³The last condition is a boring add-on. The pair f_0, f_1 should be normalized so that the ratio f_1/f_0 be an decreasing rather than an increasing function over \mathbb{R} .

PROOF. We already know that

$$\begin{array}{ccc} (1) & \Longleftrightarrow & (2) \\ \uparrow & & \downarrow \\ (4) & \Longleftarrow & (3) \implies (5) \end{array}$$

To prove the implication $(3) \Longleftarrow (5)$, note that the reality and simplicity of the roots of f_0 imply the possibility of a decomposition of type (59). Next, if two consecutive residues α_{k-1} and α_k were of different sign, then the interval (ω_{k-1}, ω_k) would contain an *even* number of roots of R . All residues α_k are therefore of the same sign, namely positive, in view of (61). \square

Remark 74. Analytic functions that map the upper half-plane into itself are well studied. They play an important role in the spectral theory of self-adjoint operators (see [2, 16, 17]). The generic representation of such a function is

$$F(z) = \alpha z + \beta + \int_{-\infty}^{+\infty} \frac{1 + \omega z}{\omega - z} d\theta(\omega) \quad (\text{Im } z \neq 0), \quad (62)$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$, and $\theta(\omega)$ is a nondecreasing function with finite limits $\theta(\pm\infty)$. The function θ has only a finite number of growth points if and only if $R = -F$ is a proper rational function.

Problem 75. Prove that the logarithmic derivative of a polynomial f with (not necessarily real) roots $\omega_1, \dots, \omega_n$ satisfies (59), where the residues α_k are equal to the multiplicities of the roots ω_k .

Problem 76. Without recourse to (62), prove that a real rational function F maps the upper half-plane into itself if and only if

$$F(z) = \alpha z + \beta + \sum_{k=1}^n \frac{\alpha_k}{\omega_k - z}, \quad \text{where } \alpha \geq 0, \beta \in \mathbb{R}, \alpha_k > 0, \omega_k \in \mathbb{R}.$$

Problem 77. Let A be a bounded self-adjoint operator on a Hilbert space \mathfrak{H} , let $u \in \mathfrak{H}$, and consider

$$R(\lambda) \equiv ((\lambda I - A)^{-1}u, u). \quad (63)$$

Prove that

$$\frac{\text{Im } R(\lambda)}{\text{Im } \lambda} = -\|(\lambda I - A)^{-1}u\|^2 \quad (\text{Im } \lambda \neq 0)$$

and therefore $-R$ maps the upper half-plane into itself.

Problem 78. Assuming that the operator of Problem 77 is continuous, prove that the corresponding function (63) admits the decomposition

$$R(\lambda) = \sum_{k=1}^{\infty} \frac{|(u, e_k)|^2}{\lambda - \omega_k},$$

where $\{e_1, e_2, \dots\}$ is an orthonormal basis consisting of eigenvectors of A , and $\{\omega_1, \omega_2, \dots\}$ are the corresponding eigenvalues.

Problem 79. Let R be a rational function that vanishes at infinity. Then it admits a Laurent series representation that converges for sufficiently large $|z|$:

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots$$

Prove that R is proper if and only if the coefficients s_0, s_1, s_2, \dots satisfy

$$s_k = \sum_{j=1}^n \alpha_j \omega_j^k \quad \text{for some } \alpha_1, \dots, \alpha_n > 0, \quad \omega_1 < \dots < \omega_n. \quad (64)$$

Problem 80. If $R \in \mathcal{R}$, then $-\frac{\operatorname{Im} R(z)}{\operatorname{Im} z} > |R'(z)|$ ($\operatorname{Im} z > 0$).

Let γ be a smooth curve lying in the upper half-plane

$$\Pi \equiv \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Define a *non-Euclidean* curve length γ by the formula $s(\gamma) \equiv \int_{\gamma} (\operatorname{Im} z)^{-1} |dz|$. This makes the half-plane Π into the *Poincaré model* of the Lobachevsky plane (see [9, 14]). The geodesics of this plane are half-circles with center on the real axis \mathbb{R} , and vertical rays. The Lobachevsky-Poincaré plane is shown on the front page in a somewhat stylized form.

Problem 81. If $R \in \mathcal{R}$, then the map $-R : \Pi \rightarrow \Pi$ decreases curve lengths on Π :

$$s(-R \circ \gamma) < s(\gamma).$$

Problem 82. Let $R \in \mathcal{R}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then the equation $R(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$

- has only real solutions if $\alpha\delta - \beta\gamma \geq 0$;
- has no more than one pair of complex conjugate solutions if $\alpha\delta - \beta\gamma < 0$;

Problem 83. View the function $p(z) = e^z$ as being analogous to a polynomial. This entire function has no roots in the right half-plane (in fact, no roots whatsoever) so can be thought of as «stable». Can one apply the results of Theorem 72, at least to some extent, to this function? Without succumbing to premature enthusiasm, consider also the function $p(z) = e^{-z}$.

9. Hankel forms

Given a real rational function⁴ R , let us associate with it a *sesquilinear form*⁵ H defined on the (infinite-dimensional) complex linear space \mathcal{P} of all polynomials by the formula

$$H(x, y) \equiv \frac{1}{2\pi i} \oint_{\gamma} R(\zeta) x(\zeta) \overline{y(\zeta)} d\zeta \quad (x, y \in \mathcal{P}). \quad (65)$$

Here $\overline{y(\zeta)} \equiv \overline{y(\overline{\zeta})}$, and γ is a positively oriented closed contour enclosing all poles of the function R . By the Cauchy residue theorem,

$$H(x, y) = \sum \operatorname{Res}_{\omega_k} (Rx\overline{y}) \quad (66)$$

where the summation is over all poles of R .

A residue is especially easy to compute at a simple pole:

$$f_0(\omega) = 0, \quad f'_0(\omega) \neq 0, \quad f_1(\omega) \neq 0 \implies \operatorname{Res}_{\omega} \left(\frac{f_1}{f_0} \right) = \frac{f_1(\omega)}{f'_0(\omega)}.$$

⁴In contrast to § 7, where a rational function was considered as a map on the real projective line, here we consider it as an analytic function of a complex variable.

⁵ $H \longleftarrow$ **H**ermite, **H**ankel, **H**urwitz.

So, if all poles $\omega_1, \dots, \omega_n$ of the function R are real and simple, then

$$H(x, x) = \sum_{k=1}^n \alpha_k |x(\omega_k)|^2, \quad \text{where} \quad \alpha_k \equiv \text{Res}_{\omega_k}(R) \in \mathbb{R}. \quad (67)$$

We thus reduced the *Hermitian form* $H(x, x)$ to a sum of squares, and formula (67) shows that the *rank* $\text{rank } H$ of this form (i.e., the total number of squares) is equal to n , the number of poles of the function R , whereas its *signature* $\text{sign } H$ (i.e., the difference between the number of positive and negative squares) is equal to $\text{Ind}_{-\infty}^{+\infty}(R)$ (Hermite 1856).

When the function R has multiple or nonreal poles, one fails to find such an expressive formula as (67). However, the qualitative connection remains valid: $\text{rank } H = \deg f_0$, $\text{sign } H = \text{Ind}_{-\infty}^{+\infty}(R)$ (Hurwitz, 1895). Let us try to penetrate the essence of this phenomenon.

Given a polynomial $g \in \mathcal{P}$, consider the subspaces

$$\mathcal{P}^g \equiv \{gu : u \in \mathcal{P}\}, \quad \mathcal{P}_g \equiv \{v \in \mathcal{P} : \deg v < \deg g\}.$$

The *restrictions* of the form H to these subspaces will be denoted by H^g and H_g , respectively. Long division of polynomials ($x = gu + v$, $\deg v < \deg g$) shows that⁶

$$\mathcal{P} = \mathcal{P}^g \oplus \mathcal{P}_g \implies \text{codim } \mathcal{P}^g = \dim \mathcal{P}_g = \deg g. \quad (68)$$

Remark 84. In addition to being a linear space, \mathcal{P} is a *commutative algebra*: its elements are multiplied according to well-known rules (see [20, 4, 5, 6]). The subspace $\mathcal{J} = \mathcal{P}^g$ is an *ideal* of the algebra \mathcal{P} :

$$x \in \mathcal{J}, y \in \mathcal{P} \implies xy \in \mathcal{J}.$$

We now assign the «inconvenient» poles of the function $R = f_1/f_0$ to a polynomial g :

- if ω is a nonreal pole of R of order ν lying, say, in the upper half plane, then it will be a root of g of multiplicity ν (then $\bar{\omega}$ will be a root of \bar{g});
- if ω is a real pole of R of order $\nu > 1$, then it will be a root of g of multiplicity $\lfloor \frac{\nu}{2} \rfloor \geq 1$.

As a result, the denominator f_0 splits (assuming that f_0 and f_1 have no factors in common) into the product

$$f_0 = g \bar{g} h, \quad (69)$$

where the roots of h are all simple and coincide with real poles of R of *odd order*.

Now assume that $x, y \in \mathcal{P}^g$, $x = g x^g$, $y = g y^g$, and let $R^g \equiv g \bar{g} R = f_1/h$. Then (65) turns into

$$H^g(x, y) = \frac{1}{2\pi i} \oint_{\gamma} R^g(\zeta) x^g(\zeta) \overline{y^g(\zeta)} d\zeta. \quad (70)$$

All poles of the function R^g are real and simple; therefore, by the argument above,

$$\text{rank } H^g = \deg h, \quad \text{sign } H^g = \text{Ind}_{-\infty}^{+\infty}(R^g). \quad (71)$$

On the other hand, the polynomial $g \bar{g}$ is nonnegative on \mathbb{R} , hence the functions R^g and R have equal signs in the neighborhood of their common real poles, and hence

$$\text{Ind}_{-\infty}^{+\infty}(R^g) = \text{Ind}_{-\infty}^{+\infty}(R). \quad (72)$$

⁶codim Y is the *codimension* of a subspace Y of a linear space X . It is equal to the dimension of a maximal subspace of $Z \subset X$ that intersects Y trivially (according to *Zorn's lemma*, such a subspace Z always exists). If $\dim X < \infty$, then $\text{codim } Y = \dim X - \dim Y$.

We are now in a position to state and prove the main theorem of this section.

Theorem 85. *The rank of the form H is equal to the number of poles of the rational function R (counted according to their multiplicities), and its signature coincides with the Cauchy index $\text{Ind}_{-\infty}^{+\infty}(R)$.*

PROOF. The following two lemmata are necessary and sufficient for our proof.

Lemma 86 (rank). $\text{rank } H = \text{rank } H^g + 2 \deg g = \deg f_0$.

PROOF. The second equality follows from (69) and (71). We now prove the first.

By the theory of quadratic and Hermitian forms (see [20, 4, 5, 6]),

$$\text{rank } H = \text{codim } \mathcal{N}, \quad \text{where } \mathcal{N} \equiv \{x \in \mathcal{P} : H(x, y) = 0, \forall y \in \mathcal{P}\}$$

is a subspace called the *kernel* of H . To describe the kernel, note that

$$x \in \mathcal{N} \iff \oint_{\gamma} R(\zeta) x(\zeta) \zeta^j d\zeta = 0 \quad (j = 0, 1, 2, \dots),$$

which means that Rx is a polynomial, x is divisible by the denominator f_0 , and $x \in \mathcal{P}^{f_0}$. Thus, $\mathcal{N} = \mathcal{P}^{f_0} \implies \text{codim } \mathcal{N} = \deg f_0$. \square

Lemma 87 (signature). $\text{sign } H = \text{sign } H^g$.

PROOF. Denote the number of positive (resp., negative) squares of the form H by $\text{pos } H$ (resp., $\text{neg } H$). The number $\text{pos } H$ coincides with the dimension of a maximal subspace where the form $H(x, x)$ is positive definite (see [20, 4, 5, 6]); the same holds for $\text{pos } H^g$, $\text{neg } H$, $\text{neg } H^g$. Lemma 86 implies that all these quantities are finite, and moreover,

$$\text{pos } H + \text{neg } H = (\text{pos } H^g + \deg g) + (\text{neg } H^g + \deg g). \quad (73)$$

Let \mathcal{P}^+ be a subspace of \mathcal{P} of dimension $\text{pos } H$, where $H|_{\mathcal{P}^+}$ is positive definite. Then the restricted form H^g is positive definite on the subspace $\mathcal{P}^g \cap \mathcal{P}^+$, so that

$$\dim(\mathcal{P}^g \cap \mathcal{P}^+) \leq \text{pos } H^g. \quad (74)$$

On the other hand,

$$\dim \mathcal{P}^+ \leq \dim(\mathcal{P}^g \cap \mathcal{P}^+) + \text{codim } \mathcal{P}^g. \quad (75)$$

(Indeed, if \mathcal{Q} is a maximal subspace of \mathcal{P}^+ that intersects \mathcal{P}^g trivially, then $\dim \mathcal{Q} = \dim \mathcal{P}^+ - \dim(\mathcal{P}^g \cap \mathcal{P}^+)$ and $\dim \mathcal{Q} \leq \text{codim } \mathcal{P}^g$.) Since $\text{codim } \mathcal{P}^g = \deg g$, formulæ (74)–(75) imply

$$\text{pos } H \leq \text{pos } H^g + \deg g \quad \text{and, analogously,} \quad \text{neg } H \leq \text{neg } H^g + \deg g,$$

but (73) implies that these must be equalities. Thus,

$$\text{sign } H^g = \text{pos } H^g - \text{neg } H^g = \text{pos } H - \text{neg } H = \text{sign } H.$$

\square

This concludes the proof of Theorem 85 as well: the statement about the rank follows from Lemma 86, and the statement about the signature follows from Lemma 87 and from equalities (71)–(72). \square

Remark 88. We now understand the influence of nonreal and multiple roots that were collected into g : they give the form H some «ballast» consisting of an equal number ($\deg g$) of positive and negative squares, which brings up the rank of H . These rather nondescript squares get filtered out when H is restricted to the ideal \mathcal{P}^g of the algebra \mathcal{P} .

Remark 89. Theorem 85 deals with $\text{Ind}_{-\infty}^{+\infty}(R)$ – not with $\text{Ind}_{\mathbb{P}\mathbb{R}}(R)$, as in § 7. However, if $\deg f_1 \leq \deg f_0$, the two indices coincide.

So far, it was convenient⁷ to consider the form H on a *complex infinite-dimensional* space. We take a more practical position now.

First of all, the form H is real: $\overline{H(x, y)} = H(\overline{x}, \overline{y})$. This implies that, for real polynomials u and v , the equality $H(u + iv, u + iv) = H(u, u) + H(v, v)$ holds. So, without loss of generality, \mathcal{P} can be assumed to be a *real* space (or algebra).

Secondly, denote for simplicity $f \equiv f_0$ and recall that $\mathcal{P} = \mathcal{P}_f \oplus \mathcal{P}^f$ (relation (68)) as well as $\mathcal{P}^f = \mathcal{N}$ (the proof of Lemma 86). This implies that

$$H(x_f + x^f, y_f + y^f) = H_f(x_f, y_f) \quad (x_f, y_f \in \mathcal{P}_f; \ x^f, y^f \in \mathcal{P}^f). \quad (76)$$

Lemma 90. $\text{rank } H = \text{rank } H_f$, $\text{sign } H = \text{sign } H_f$

PROOF. With \mathcal{P}^+ as in the proof of Lemma 87, formula (76) yields

$$x \in \mathcal{P}^+, \ x \neq 0 \implies H(x, x) = H_f(x_f, x_f) > 0 \implies x_f \neq 0.$$

So, the projector $x \mapsto x_f$ to the first component of the direct sum⁸ $\mathcal{P}_f \oplus \mathcal{P}^f$ is injective on \mathcal{P}^+ , hence does not decrease its dimension. Therefore $\text{pos } H_f \geq \text{pos } H$, while the reverse inequality is obvious. Analogously, $\text{neg } H_f = \text{neg } H$. \square

Thus, we can consider the form H on the *finite-dimensional* space \mathcal{P}_f . However, \mathcal{P}_f , unlike \mathcal{P} , is not an algebra. On the other hand, the form H_f is necessarily *nondegenerate*:

$$x \in \mathcal{P}_f, \ H_f(x, y) = 0, \ \forall y \in \mathcal{P}_f \implies x = 0.$$

Finally, consider a basis in \mathcal{P}_f and write the form H_f in its canonical form.

Let the basis consist of monomials ζ^j ($j = 0, 1, \dots, n-1$; $n \equiv \deg f$, $f \equiv f_0$). Then (65) shows that

$$x(\zeta) = \sum_{j=0}^{n-1} \xi_j \zeta^j \implies H_f(x, x) = \sum_{i,j=0}^{n-1} s_{i+j} \xi_i \xi_j, \quad (77)$$

where

$$s_k \equiv \frac{1}{2\pi i} \oint_{\gamma} R(\zeta) \zeta^k d\zeta \quad (k = 0, 1, 2, \dots). \quad (78)$$

If the rational function R is expanded into a series

$$R(\zeta) = s_{-m} \zeta^{m-1} + \dots + \frac{s_0}{\zeta} + \frac{s_1}{\zeta^2} + \dots \quad (m = \deg f_1 - \deg f_0) \quad (79)$$

(which converges absolutely for large values of $|\zeta|$ if $m \leq 1$), then, substituting (79) into (78) and integrating term by term, we verify that the coefficients s_k ($k \geq 0$) in (79) satisfy conditions (78).

An unusual property of (77) is that each of its coefficients s_{i+j} depends only on the *sum* of its indices. Quadratic forms of this type are called *Hankel forms*.

The next theorem adds to our already large collection of statements that characterize rational functions from the class \mathcal{R} :

⁷Does the reader see why?

⁸Simply, this is division of the polynomial x with quotient f and remainder x_f .

Theorem 91. *Let*

$$R(\zeta) = \frac{f_1(\zeta)}{f_0(\zeta)} = \frac{s_0}{\zeta} + \frac{s_1}{\zeta^2} + \cdots \quad (\deg f_1 < \deg f_0 = n).$$

Then $R \in \mathcal{R}$ if and only if

$$s_0 > 0, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} > 0, \quad \begin{vmatrix} s_0 & \cdots & s_{n-1} \\ \vdots & \ddots & \vdots \\ s_{n-1} & \cdots & s_{2n-2} \end{vmatrix} > 0. \quad (80)$$

PROOF. The condition $R \in \mathcal{R}$ is equivalent to the positive definiteness of the form (77), which is by Sylvester's criterion equivalent to (80). \square

Remark 92. It would be interesting to find out how the determinants in (80) can be expressed directly in terms of the coefficients of the polynomials f_1 and f_0 . We will take up this question in the next section.

Problem 93. *Prove that \mathcal{P} is a principal ideal domain. The latter means [20, 4, 5, 6] that any ideal $\mathcal{J} \subset \mathcal{P}$ (see Remark 84) is generated by some polynomial $g \in \mathcal{P}$: $\mathcal{J} = \mathcal{P}g$.*

Problem 94. *Let $f, g \in \mathcal{P}$ and let $d = \gcd(f, g)$, $k = \text{lcm}(f, g)$. Prove that $\mathcal{P}^k = \mathcal{P}^f \cap \mathcal{P}^g$ and $\mathcal{P}^d = \mathcal{P}^f + \mathcal{P}^g$. This explains the real meaning and role of the terms «greatest common factor» and «least common multiple».*

Problem 95. *Under the assumptions of Problem 91, suppose the following inequalities hold as well:*

$$s_1 > 0, \quad \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix} > 0, \quad \begin{vmatrix} s_1 & \cdots & s_n \\ \vdots & \ddots & \vdots \\ s_n & \cdots & s_{2n-1} \end{vmatrix} > 0. \quad (81)$$

What additional properties of the function R follow?

Problem 96. *Prove the Borchardt-Jacobi Theorem: Let f be a real polynomial of degree n with (complex) roots $\lambda_1, \dots, \lambda_n$, and let⁹*

$$s_k = \lambda_1^k + \cdots + \lambda_n^k \quad (k = 0, 1, 2, \dots).$$

Then the number of positive squares of the form $\sum_{i,j=0}^{n-1} s_{i+j} \xi_i \xi_j$ coincides with the number of distinct roots of the polynomial f , and the number of its negative squares with the number of distinct complex conjugate pairs of roots.

Hint: One can apply the results of this and previous Sections to the logarithmic derivative of the polynomial f , but a direct argument is also possible.

Problem 97. *Derive Newton's identities that connect the Newton sums s_0, s_1, \dots with the coefficients a_0, a_1, \dots, a_n of the polynomial f from the previous problem.*

Hint: Prove the relation

$$\frac{na_0\zeta^{n-1} + \cdots + a_{n-1}}{a_0\zeta^n + \cdots + a_n} = \frac{s_0}{\zeta} + \frac{s_1}{\zeta^2} + \frac{s_2}{\zeta^3} + \cdots$$

and make use of it.

⁹The quantities being defined here are called Newton sums. They are *symmetric* functions of the roots λ_j ; therefore they can be found without knowing the actual roots (see [7]).

Problem 98. As in Problem 66, let $R(\lambda) \equiv \varphi((\lambda I - A)^{-1}u)$, where $A : \mathfrak{B} \rightarrow \mathfrak{B}$ is a bounded linear operator on a Banach space \mathfrak{B} , $u \in \mathfrak{B}$, $\varphi \in \mathfrak{B}'$. Prove that

$$R(\lambda) = \frac{s_0}{\lambda} + \frac{s_1}{\lambda^2} + \frac{s_2}{\lambda^3} + \cdots, \quad \text{where } s_k = \varphi(A^k u) \quad (k = 0, 1, 2, \dots).$$

Problem 99. As in problem 77, let $R(\lambda) \equiv ((\lambda I - A)^{-1}u, u)$, where $A : \mathfrak{H} \rightarrow \mathfrak{H}$ is a bounded self-adjoint operator on a Hilbert space \mathfrak{H} . Prove that the Hankel forms

$$\sum_{i,j=0}^{n-1} s_{i+j} \xi_i \xi_j \quad (n = 1, 2, \dots), \quad \text{where } s_k = (A^k u, u) \quad (k = 0, 1, \dots)$$

are nonnegative definite. In which case are they positive definite?

Problem 100. Let A be a real $n \times n$ -matrix whose elements and minors are all positive¹⁰. Let s_k ($k = 0, 1, 2, \dots$) be the $(1, 1)$ entry of the k th power A^k of the matrix A . Prove that inequalities (80) and (81) hold.

Hint: Find formulæ connecting the determinants in (80)-(81) with the minors of A .

10. Liénard-Chipart criterion

We now attempt to combine the ideas of the preceding three sections.

The rank and the signature of a Hankel form are *not* projective invariants, since they do not account for a possible pole of the function $R = f_1/f_0$ at the point ∞ . However, if $\deg f_1 \leq \deg f_0$, then $R(\infty) \neq \infty$ and $\text{Ind}_{\mathbb{P}\mathbb{R}}(R) = \text{Ind}_{-\infty}^{+\infty}(R)$. Moreover, if, as in § 7,

$$\Phi \in \mathbf{SL}(2, \mathbb{R}), \quad \Phi : \omega \mapsto \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \quad (\alpha\delta - \beta\gamma = 1),$$

then

$$(\Phi \circ R)(z) = \frac{\alpha f_1(z) + \beta f_0(z)}{\gamma f_1(z) + \delta f_0(z)},$$

and

$$\deg(\alpha f_1 + \beta f_0) = \deg(\gamma f_1 + \delta f_0) = \max\{\deg f_1, \deg f_0\}$$

for all $\Phi \in \mathbf{SL}(2, \mathbb{R})$, except for exactly two elements: Φ_0 when $\Phi_0 \circ R$ has a zero at the point $z = \infty$, and Φ_∞ if $z = \infty$ is a pole of $\Phi_\infty \circ R$.

Lemma 101. *If*

$$R(z) \equiv \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{c_0 z^n + c_1 z^{n-1} + \cdots + c_n} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots \quad (c_0 \neq 0), \quad (82)$$

*then*¹¹

$$\nabla_{2k} \equiv \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{k-1} & c_k & \cdots & c_{2k-1} \\ b_0 & b_1 & b_2 & \cdots & b_{k-1} & b_k & \cdots & b_{2k-1} \\ 0 & c_0 & c_1 & \cdots & c_{k-2} & c_{k-1} & \cdots & c_{2k-2} \\ 0 & b_0 & b_1 & \cdots & b_{k-2} & b_{k-1} & \cdots & b_{2k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_0 & c_1 & \cdots & c_k \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_k \end{vmatrix} = c_0^{2k} \det [s_{i+j}]_0^{k-1} \quad (83)$$

($k = 1, 2, \dots, n$; as in § 6, we set $c_j = b_j = 0$ for $j > n$).

¹⁰Such matrices are called totally positive; for details, see [10].

¹¹Lo and behold the Hurwitz matrix!

PROOF. First interchange the rows of the determinant in (83) to obtain

$$\nabla_{2k} = \begin{vmatrix} c_0 & c_1 & c_2 & \dots & c_{k-1} & & c_k & \dots & c_{2k-1} \\ 0 & c_0 & c_1 & \dots & c_{k-2} & & c_{k-1} & \dots & c_{2k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_0 & \checkmark & c_1 & \dots & c_k \\ 0 & 0 & 0 & \dots & b_0 & \checkmark & b_1 & \dots & b_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & b_0 & b_1 & \dots & b_{k-2} & & b_{k-1} & \dots & b_{2k-2} \\ b_0 & b_1 & b_2 & \dots & b_{k-1} & & b_k & \dots & b_{2k-1} \end{vmatrix}. \quad (84)$$

This does not change the sign of ∇_{2k} . Indeed, lower the k th and $(k+1)$ st rows¹² to their initial positions in (83). This will require an *even* number of transpositions. The next pair of rows will then meet, the lowering operation will be applied to them, and so on.

Now let us establish a connection between b, c and s . Multiplying (82) by the denominator and equating coefficients, we get:

$$b_j = \sum_{i=0}^j c_{j-i} s_{i-1} \quad (j = 0, 1, 2, \dots). \quad (85)$$

These formulæ suggest by themselves what to do next, namely, to eliminate the entries in the lower left corner of the determinant (84). From each $k+j$ th row ($j = 1, 2, \dots, k$) subtract rows $k-j+1, k-j+2, \dots, k$ multiplied by $s_{-1}, s_0, \dots, s_{j-2}$, respectively. As a result, we get zeros down and to the left, and the entry

$$d_{ij} \equiv b_{i+j-1} - c_{i+j-1} s_{-1} - \dots - c_i s_{j-2} = c_{i-1} s_{j-1} + \dots + c_0 s_{i+j-2} \quad (i, j = 1, 2, \dots, k)$$

in position $(k+i, k+j)$. In matrix form, this will look as follows:

$$\begin{vmatrix} d_{11} & d_{12} & \dots & d_{1k} \\ d_{21} & d_{22} & \dots & d_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{k1} & d_{k2} & \dots & d_{kk} \end{vmatrix} = \begin{vmatrix} s_0 & s_1 & \dots & s_{k-1} \\ s_1 & s_2 & \dots & s_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \dots & s_{2k-2} \end{vmatrix} \cdot \begin{vmatrix} c_0 & c_1 & \dots & c_{k-1} \\ 0 & c_0 & \dots & c_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_0 \end{vmatrix}.$$

□

Remark 102. The quantities $\nabla_{2k} = \nabla_{2k}(R)$ are invariants under the action of the group $SL(2, \mathbb{R})$, since the structure of determinants (83) implies that $\nabla_{2k}(R + d) = \nabla_{2k}(R)$ ($d = \text{const}$) and $\nabla_{2k}(-\frac{1}{R}) = \nabla_{2k}(R)$ due to Lemmata 7.1–7.3.

Remark 103. ∇_{2n} coincides, up to a sign, with the *resultant* of the polynomials f_1 and f_0 . Hence

$$\nabla_{2n} \neq 0 \iff \gcd(f_0, f_1) = 1$$

More about the resultant and other *symmetric polynomials* can be found in [7]; this topic is worth studying *per se*, but we do not need to invoke external results to justify the fact we just stated. Indeed, the inequality $\nabla_{2n} \neq 0$ together with Lemma 101 imply that the rank of the corresponding Hankel form H , which coincides with the total number of the poles of the function f_1/f_0 , is at least $\deg f_0$, hence the fraction f_1/f_0 is in lowest terms.

¹²They are marked by \checkmark in (84).

Lemma 104. *With the notation of Lemma 101,*

$$\nabla_2 > 0, \nabla_4 > 0, \dots, \nabla_{2n} > 0 \implies \frac{\operatorname{Im} R(z)}{\operatorname{Im} z} < 0 \quad (\operatorname{Im} z \neq 0).$$

If n is the number of poles of the rational function R , then the converse holds as well.

PROOF. If $b_0 = 0$, apply Lemma 101, Theorem 91, and Theorem 72. If $b_0 \neq 0$, apply the same results to the function $R(z) - s_{-1}$. \square

We already noticed that the determinants ∇_{2k} ($k = 1, 2, \dots, n$) are the even leading principal minors of the $2n \times 2n$ matrix built exactly as the Hurwitz matrix \mathcal{H}_p from § 6. This is not just a superficial similarity. Let us go back to the main object of our study, viz., the real polynomial

$$p(z) \equiv a_0 z^n + a_1 z^{n-1} + \dots + a_n \quad (a_0 > 0) \quad (86)$$

and let us rewrite it as

$$p(z) = g_0(z^2) + z g_1(z^2).$$

The polynomials g_0 and g_1 so defined are very much reminiscent of f_0 and f_1 , which first appeared in § 3 and were studied in detail in §§ 5, 6. If the degree n is odd $n = 2m + 1$, then

$$\begin{aligned} g_0(w) &= a_1 w^m + a_3 w^{m-1} + \dots + a_{2m+1}, \\ g_1(w) &= a_0 w^m + a_2 w^{m-1} + \dots + a_{2m}; \end{aligned}$$

if $n = 2m$, then

$$\begin{aligned} g_0(w) &= a_0 w^m + a_2 w^{m-1} + \dots + a_{2m}, \\ g_1(w) &= a_1 w^{m-1} + a_3 w^{m-2} + \dots + a_{2m-1}. \end{aligned}$$

In either case, $\deg g_0 \geq \deg g_1$, and the minors $\nabla_{2k} = \nabla_{2k}(g_1/g_0)$ and the minors η_k of the Hurwitz matrix \mathcal{H}_p (see § 6) are connected as follows¹³:

$$\nabla_{2k} = \begin{cases} \eta_{2k} & (k = 1, \dots, m), & \text{if } n = 2m + 1 \\ a_0 \eta_{2k-1} & (k = 1, \dots, m), & \text{if } n = 2m. \end{cases} \quad (87)$$

If, as usual, $R \equiv g_1/g_0$, then

$$p(z) = 0 \iff R(z^2) = -\frac{1}{z}. \quad (88)$$

We are now ready to prove the Liénard-Chipart criterion, which was mentioned in § 6:

Theorem 105 (Liénard–Chipart). *A polynomial (86) is stable if and only if*

$$a_n > 0, a_{n-1} > 0, a_{n-2} > 0, \dots \quad (89)$$

$$\eta_{n-1} > 0, \eta_{n-3} > 0, \eta_{n-5} > 0, \dots \quad (90)$$

PROOF. Necessity: follows from the Stodola condition and the Hurwitz criterion. Sufficiency: 1. Since the minors are positive, Lemma 104 and formula (87) imply

$$\frac{\operatorname{Im} R(z)}{\operatorname{Im} z} < 0 \quad (\operatorname{Im} z \neq 0)$$

¹³Note that in both cases the *last* ∇_{2m} corresponds to the *second-last* η_{n-1} , which, as we know from § 5, «guards» the imaginary axis!

So, if $p(z) = 0$, $\operatorname{Im} z \neq 0$, then (88) and (90) give

$$\frac{\operatorname{Im} R(z^2)}{\operatorname{Im} z^2} = \frac{\operatorname{Im} z}{|z|^2 \operatorname{Im} z^2} = \frac{1}{2|z|^2 \operatorname{Re} z} < 0.$$

Thus, by (90), the roots of the polynomial p lie in the union of the open left-hand plane and the real positive half-line.

2. Since the coefficients are positive, the polynomial p cannot have any roots on the nonnegative side of the real axis. \square

Remark 106. If we drop (89) but keep (90) in Theorem 105, then some roots of p may cross over into the right half-plane, but in that case *they must stay on the positive real half-line*. From the stability point of view, this behavior is very interesting, since it means that the fixed point does not bifurcate into a limit cycle (these bifurcations are described in detail in [21]).

Problem 107. *For a polynomial discussed in Remark 106 whose roots with positive real part cannot leave the real axis, it is natural to expect that these roots stay simple (a simple root cannot leave the real axis other than by coalescing with another root). Prove that this is indeed so, i.e., that the positive roots of a polynomial satisfying (90) are simple.*

Afterword

Half a page is still left – it would be sinful to leave it blank. Let us draw conclusions.

In stability theory, the Routh-Hurwitz problem, which was considered in these «lecture notes» from many points of view, certainly does not play a role commensurate with the attention we devoted to it; more precisely, it does not yet play a role it is destined for. Destined by whom? I don't know. Still, I believe that intrinsically beautiful mathematical constructs must be necessarily connected to the understanding and explanation of the real world that surrounds us. If you wish, you may label this a mathematical religion of sorts; I think many mathematicians, perhaps most, are such conscious or unconscious believers. It is impossible to imagine that the unreal world studied by mathematics has been created by human intellect; mathematicians do not invent theorems and theories but *discover* them. And this ideal world of mathematical constructions always turns out to parallel the «real» one; remarkably, its researchers are driven neither by «logic», as laymen think, nor by «applied» needs, nor even by acquired experience and knowledge, both certainly indispensable. They are driven by an irrational, strange intuition that lets them feel that intrinsic beauty and harmony, just as our senses can feel warmth and determine its source. It is true that the mathematician Hurwitz was «handed» a problem by the turbine engineer Stodola, but Hurwitz took up and solved that problem not to help Stodola build his turbines. Well, he would have not taken it up just for that. Such is indeed the relationship between mathematics and its various «applications»: the latter are sources of problems for the former, and the origin of these problems is a certain *a priori* guarantee of the harmony and beauty to be found there, and of the progress in mathematics that their solutions must bring about.

Returning to the beginning of our scholia, let me venture an opinion: the sad fact that the «Routh-Hurwitz problem» is not much in demand does not mean that it is only of academic interest. This should rather mean that there is a hidden door behind which there may be lots of interesting stuff. Just as in the instructive story of “The golden key”¹⁴. :–)

¹⁴“The golden key” by Alexei Tolstoi, a famous Russian adaptation of the book “The adventures of Pinocchio” by Carlo Collodi [translators’ remark].

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*Comments.*¹⁵ The books [11, 27, 18] offer a thorough treatment of the subject. The book [27] is most elementary and detailed. The monograph [11] remains one of the best matrix theory books in the world. The paper [18] contains an exhaustive review of classical works on stable polynomials. The textbook [8] states the Hurwitz theorem and the «Mikhailov criterium». However, the relevant sections of this overall good book are in my opinion not quite satisfactorily, and it is better to use [27].

The monographs [1, 7] are devoted to the Routh-Hurwitz problem for entire functions.

The books [22, 23, 24, 25, 26] of G. Pólya mentioned in the Introduction are not directly related to our topic of stable polynomials, but their reading is useful for every beginning mathematician.

The book [12] is outdated and is written «for dummies». However, the now forgotten language APL, which was created by Kenneth Iverson not exactly as a practical programming language but rather as a notation system for mathematical algorithms, is *per se* interesting to a mathematician.

¹⁵Many of the author’s original references, in Russian or translated into Russian, were replaced by the corresponding references in English [translators’ remark].